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Matrix models and the gravitational interaction

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Abstract

The superstring theory is now regarded as the most promising candidate for the unification of the Standard Model and gravity, and this field has been rigorously investigated. However, we have seen a setback of the perturbative analysis of the superstring theory, because it has so many vacua that we have no way to determine which are the true ones. In order to remedy this problem and thus for the superstring theory to have a power to predict our real four-dimensional world, we need the nonperturbative formulation of the superstring theory.

In the late 1990's, our understanding of the nonperturbative aspects of the superstring theory have been greatly deepened. Especially, the large-N (N is the size of the matrices) reduced models have been proposed as the nonperturbative formulation of the superstring theory. One of the most promising candidates is the IIB matrix model, which is defined by the dimensional reduction of the ten-dimensional $\mathcal{N} = 1$ super-Yang-Mills theory to zero dimension. It has been conjectured that this model resurrects the behavior of the string theory at the large-N limit. There have been a lot of interesting discoveries of the IIB matrix model, such as the dynamical generation of the four-dimensional spacetime and the interpretation of the diffeomorphism invariance.

On the other hand, there are a lot of problems to surmount, if a large-N reduced model is to be an eligible framework to unify the gravitational interaction. Firstly, it is still an enigma how we can realize the local Lorentz invariant matrix model. In addition, we need to understand how we can describe the curved spacetime more manifestly, in terms of a large-N reduced model.

This thesis discusses several attempts to address these issues concerning the gravitational interaction. This thesis is based on the following works [26, 38, 46, 60].

Contents

1 Introduction

One of the main themes in the elementary particle physics is unification. There are four kinds of interaction in the nature; the weak interaction, the strong interaction, the electromagnetic interaction and the gravitational interaction. In 1967, Glashow, Weinberg and Salam succeeded in the unification of the weak and electromagnetic interaction in terms of the $SU(2) \times U(1)$ gauge group. The ensuing success is the advent of the "Standard Model", described by the $SU(3) \times SU(2) \times U(1)$ gauge group. The Standard Model is striking in the sense that it describes all the experimental phenomena by setting the 18 parameters in the model.

However, there are two serious drawbacks to the Standard Model. First is that we cannot explain theoretically how these 18 parameters are fixed. Namely, we have to rely on the experimental data in setting these parameters. Second is that the Standard Model does not unify the gravitational interaction.

Now, the superstring theory is regarded as the most promising candidate to resolve these two drawbacks. The superstring theory regards not a zero-dimensional point but a one-dimensional string with the length $l_p = 10^{-33}$cm as the fundamental object. Here, l_p is called "the Planck Scale", which is the fundamental scale in the superstring theory. The superstring theory incorporates not only the gauge particles but also the gravitons in its oscillation modes. There is an infinite tower of the mass level in the oscillation modes. The massless modes of the open string, which has two ends, are regarded as the gauge particle. On the other hand, the massless modes of the closed string, which constitutes a closed circle like a rubber band, gives the spin-2 graviton. In this sense, the superstring theory is thought to include not only the Standard Model but also the gravitational theory in the low-energy limit, where the massive modes of the superstring theory are ignored. The superstring theory is fascinating in that it incorporates no free parameters. It is in contrast to the Standard Model, which suffers the problem of a plethora of free parameters. The superstring theory has a possibility to explain the gauge group of the Standard Model, the number of the generations of the quarks, the mass of the Higgs particle, quarks or leptons, the dimensionality of our spacetime $\cdots$, from a theory without any parameter.

We have seen the so-called "first string boom" in the early 1980's, in which the perturbative aspects of the superstring theory have been elucidated. The striking discovery in 1984 is that the superstring theory is free from the divergence of the gravitational energy. Physicists have been so far puzzled by the UV divergence in the quantization of the gravitational interaction. This conundrum has been resolved by the superstring theory which alleviates this divergence. Moreover, it has also been known that there are only five kinds of superstring theory well-defined in the ten dimensions. They are the type I, type IIA, type IIB, $SO(32)$ heterotic and $E_8 \times E_8$ heterotic superstring theory. Especially, the $E_8 \times E_8$ heterotic superstring theory is propitious for the immersion of the Standard Model in the superstring theory. The exceptional Lie algebra E_8 includes the Lie algebra $SU(3) \times SU(2) \times U(1)$, which is the gauge symmetry of the Standard Model, as its subalgebra.

The superstring theory is defined in the ten-dimensional spacetime, but this is not so problematic because it is a theory of gravity. The spacetime is regarded as being given not a priori but dynamically from the classical solution of the theory. It is now believed that the superfluous six dimensions are compactified at an extremely small size at the incunabula of the universe. Mathematically, the six superfluous dimensions are compactified by the Calabi-Yau manifold.

However, the superstring theory suffers a serious setback. Since the superstring theory is defined only perturbatively, there is no telling which is the true vacuum. While there are only five well-defined superstring theories, there are a myriad of ways to compactify the ten-dimensional superstring theory into the four dimensions and thus we are overwhelmed by the infinite vacua. This means that the superstring theory does not have the ability to predict our four-dimensional world, and we cannot tell whether the superstring theory dynamically generates the Standard Model. In order to surmount this difficulty, we definitely need the constructive definition (i.e. the definition without the perturbation).

The so-called "first string boom" was over at the end of the 1980's. However, there were important discoveries at this period about the noncritical string theory and the bosonic matrix model (the extensive review can be found in [1,2]). Distler and Kawai [72] succeeded in the quantization of the non-critical string theory up to one dimension via the conformal gauge. In addition, there is a striking relation between the bosonic string theory and the matrix model, proposed by F. David [70]. For the simplest zero-dimensional non-critical string, the corresponding matrix model is a one-matrix model described by a simple $N \times N$ hermitian matrix. The correspondence has been found for the central charge $c = 1 - \frac{6}{m(m+1)}$ (for $m = 2, 3, 4, \cdots$), in which the string corresponds to the multi-matrix model. The path integral of the worldsheet for all genera has been approximated by the random triangulation of the worldsheet. This

has led us to identify the Feynman rule with that of the one-matrix model. In this way, we have seen the formal correspondence between the matrix model and the string theory. Even more crucial is that the matrix model can be solved nonperturbatively. Brezin and Kazakov [74] solved the matrix model by means of the orthogonal polynomial method. They extracted the nonperturbative aspects of the bosonic matrix model via the differential equation called Painlevè equation. They calculated the parameter called the string susceptibility, and found that this agrees with the calculation of Distler and Kawai [72] for the noncritical string.

We have failed in extending this idea to the "super"string, and this progress is limited to the bosonic string. This is due to the same difficulty as we face in putting the chiral fermion onto the lattice. Thus, these discoveries, per se, do not give a technical clue to the constructive definition of the superstring theory. The 'state-of-the-art' matrix models (such as the IIB matrix model) do not inherit the same techniques as the old matrix model. Nevertheless, these discoveries serve as an important touchstone for the legitimacy of the large-N reduced models as the constructive definition of the string theory.

In the late 1990's we have seen the so-called "second string boom", in which we grasped the nonperturbative aspects of the superstring theory. During the second string boom, Polchinski discovered the soliton-like object called the D-brane. The discovery of the D-brane led to the idea of the two kinds of the duality of the string theory. First is the T-duality, which is the duality between the large and small radius of the compactification. Second is the S-duality, which relates the strong coupling and the weak coupling. It is now believed that the five ten-dimensional superstring theories are related with one another through the T/S duality. This is an illuminating discovery, in the sense that it is reminiscent of the more fundamental theory whose certain limit might reproduce these five superstring theories.

Another big progress is the proposal for the constructive definition of the superstring theory via the large-N reduced model. Here, the 'large-N reduced model' means the model defined by the $N \times N$ hermitian matrices. In 1996, Banks, Fischler, Shenker and Susskind proposed the matrix model defined by the dimensional reduction of the ten-dimensional super Yang-Mills theory to one dimension $-$ the so-called BFSS model [4]. This matrix model is more related to the type IIA superstring theory, and the effect of the type IIA supergravity is induced by the one-loop effect.

Other attempts are the IIB matrix model (the IKKT model) [5] and the matrix string theory [6], which are defined by the dimensional reduction of the ten-dimensional super Yang-Mills theory to the zero and two dimensions, respectively. Especially, the IIB matrix model is now regarded as one of the most promising candidates for the constructive definition of the superstring theory (this model is extensively reviewed in [16, 75]). As its nomenclature shows, this model is deeply related to the type IIB superstring theory. The simplest and the most direct correspondence is that the IIB matrix model is defined by the matrix regularization of the Green-Schwarz action of the type IIB superstring theory; namely by replacing the Poisson bracket with the commutator of the $N \times N$ matrix. In addition, it has been shown [7] that the Wilson loops satisfy the string field equations of motion for the type IIB superstring in the light-cone gauge. The prominent feature of this model is that it does not include any free parameter. The overall coefficient which is an artifact of the coupling constant before being reduced is absorbed by a simple field redefinition.

There have been hitherto a lot of progresses with respect to the IIB matrix model. First is the relation to the gravitational interaction. The most fundamental feature is the spacetime $\mathcal{N} = 2$ supersymmetry [5] when we regard the eigenvalues of the bosonic matrices as the spacetime coordinate. This urges us to interpret the eigenvalues as the spacetime coordinate. In addition, the graviton and dilaton exchange has been calculated in [5]. The interpretation of the general coordinate invariance is proposed in [14], in which the permutation invariance of the eigenvalues is identified with the general coordinate invariance of the low-energy effective action.

Second is the relation to the noncommutative geometry, which has been widely investigated since Seiberg and Witten elucidated the relationship with the superstring theory [18]. By expanding the IIB matrix model around the classical background, it reproduces the noncommutative Yang-Mills theory [17].

Third is the dynamical generation of the four-dimensional spacetime. In [11], it has been found that the Hausdorff dimension of the eigenvalue distribution is four. In [36], Nishimura and Sugino proposed the breakdown of the Lorentz symmetry from $SO(10)$ to $SO(4)$ by means of the third order of the Gaussian expansion. Their analysis has been extended to the seventh order in [39, 48].

The discovery of the IIB matrix model implies the trilateral relationship of the three important notions in the elementary particle physics; the superstring theory, the quantum field theory and the matrix model. The discovery of the relationship between the one-matrix model and the bosonic string theory is another good example to relate these notions. In the course of the "second string boom" we have other important

discoveries for this trilateral relationships. In 1997, Maldacena advocated the duality between the type IIB superstring theory on the $AdS_5 \times S^5$ spacetime and the $\mathcal{N} = 4$, four-dimensional super Yang-Mills theory. This is the so-called "AdS/CFT correspondence" [9, 12, 15]. This correspondence has been conjectured through the coincident symmetries of these two theories. Since the proposal of Maldacena, the AdS/CFT correspondence has been rigorously investigated by many authors (among them is the author's work [32], while we do not delve into this work in this thesis). Another important discovery is the so-called Dijkgraaf-Vafa duality [41, 43, 45]. This is a duality between the $\mathcal{N} = 1$ four-dimensional super Yang-Mills theory and the one-matrix model, which used to attract much attention in relation to the bosonic string theory in the early 1990's. This discovery reinstated the one-matrix model into many authors' attention. At first [41, 43, 45], this relationship is understood with the intervention of the topological string theory. However, Cachazo, Douglas, Seiberg and Witten [47, 49, 53] explained this duality directly through the super Yang-Mills theory without topological string theory. They elucidated that these two theories comply with the same Schwinger-Dyson equation, employing the notion of the chiral ring and the Konishi anomaly [67, 68]. Moreover, in [54, 58], the direct relationship between the large-N reduction and the Dijkgraaf-Vafa duality has been found. In this way, in the course of the "second string boom", many interesting discoveries, including the IIB matrix model, have been unraveled. In the future, we will surely have a synergy in the studies of these three germane areas.

While the IIB matrix model has a lot of interesting properties, there is a plenty of room to investigate. The more manifest correspondence with the gravitational interaction is definitely a $64,000 question. If the IIB matrix model or its extensions are to be the eligible frameworks to unify the gravitational interaction, we must take seriously the relation to the gravitational interaction. While several important evidences have been found, there are a lot of questions to clarify. It has been an enigma how we can describe the local Lorentz symmetry in terms of the large-N reduced models. In addition, the definition of the IIB matrix model relies heavily on the flat background. Especially, the IIB matrix model incorporates the background of the flat space, not the curved space. This prohibits the perturbation around the curved space background.

In this thesis, we discuss some of the attempts [26, 38, 46, 60] to address these issues. This thesis is organized as follows. In Section 2, we give a brief review of the IIB matrix model. Especially, we focus on the development of the IIB matrix model in relation to the gravitational interaction. Moreover, we have a careful look at the alterations of the IIB matrix model that incorporate a curved-space background, in relation to some of the author's works [46, 60].

In Section 3, we review the author's works about the supermatrix model [26, 46], based on the $osp(1|32, R)$ super Lie algebra. The studies of the supermatrix models give a rich perspective for the generalization of the IIB matrix model in relation to the gravitational interaction. We start with investigating the correspondence of the nongauged $osp(1|32, R)$ model with the IIB matrix model, paying attention to the supersymmetry. We next consider the so-called "gauged version", namely the model whose Lorentz symmetry and the gauge symmetry are mixed together. This is an important idea in realizing the matrix model equipped with the local Lorentz symmetry, because the eigenvalues of the bosonic matrices are identified with the spacetime in the IIB matrix model. We next go back to the nongauged version of the $osp(1|32, R)$ supermatrix model including the mass term [46]. It has been known that the IIB matrix model with the tachyonic mass term, whose review we give in Section 2, incorporates the curved-space background because the trivial commutative background is destabilized. We focus on the similarity of such models and the massive supermatrix model.

In Section 4, we elaborate on the idea of the "gauged matrix model" without the supermatrix model [38]. Especially, we allocate the odd-rank matrices of the ten dimensions for the matter field, and the even-rank matrices to the parameters of the local Lorentz transformation. When we mix the Lorentz symmetry and the gauge symmetry, the model must inevitably include the higher-rank fields in order for the action to be invariant and for the algebra of the symmetry to close with respect to the commutator. This is a good news in that the rank-3 fields are identified with the spin connection in the curved space. We consider the explicit identification of the matrices with the differential operator, and clarify that the bosonic part actually reduces to the Einstein gravity in the low-energy limit. We also discuss the structure of the $\mathcal{N} = 2$ supersymmetry.

In Section 5, we study the stability of the fuzzy sphere background in the quantum sense. While it is easy to discuss the stability of the curved-space background (all we have to do is just to plug the solution into the action), it is by no means easy to discuss the stability in the quantum sense. In this section, we address this question via the heat bath algorithm of the Monte Carlo simulation. We focus on the simplest case; namely the bosonic three-dimensional IIB matrix model with the Chern-Simons term. We

find that this bosonic matrix model has a first-order phase transition between the Yang-Mills phase (in which the quantum effect is large and the fuzzy sphere classical solution is unstable) and the fuzzy sphere phase (in which the model is subject to the meager quantum effect and the fuzzy sphere is stable), as we change the radius of the fuzzy-sphere solution (equivalently, the coefficient of the Chern-Simons term). Moreover, we find that this model has a one-loop exactness in the fuzzy sphere phase in the large-N limit. The latter result is especially exciting, in the sense that this helps the analysis of the dynamical generation of the gauge group.

Section 6 is devoted to the conclusion and the future outlook of the work.

In Appendix A, we summarize the notation of this thesis. We give the detailed notation of the gamma matrices and the supermatrices. In addition, we give a proof of the miscellaneous formulae of the gamma matrices.

In Appendix B, we give the calculation of the Seeley-de-Witt coefficients, which plays an important role in Section 4.

In Appendix C, we give the detailed recipe for the heat bath algorithm of the matrix model. We start with the quadratic one-matrix model in full detail, because this gives the key idea of the simulation of the IIB matrix model with/without the Chern-Simons term. We nextly discuss the simulation of the quartic one-matrix model. Finally, we discuss the heat bath algorithm for the IIB matrix model with/without the Chern-Simons term.

2 Brief review of the IIB matrix model

In this section, we give a brief review of the IIB matrix model [5]. As we have mentioned in the introduction, the IIB matrix model is regarded as the most promising candidate for the constructive definition of the superstring theory. This model is also called "the IKKT model", where IKKT is the acronym for the authors of [5]; Ishibashi, Kawai, Kitazawa and Tsuchiya.

2.1 Definition and symmetry of the IIB matrix model

The IIB matrix model is defined by the dimensional reduction of the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory to zero dimension as

$$S = -\frac{1}{g^2} Tr \left(\frac{1}{4} \sum_{\mu,\nu=0}^{9} [A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2} \sum_{\mu=0}^{9} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right). \tag{2.1}$$

Here, A_μ are the ten-dimensional bosonic vector, and ψ is the ten-dimensional Majorana-Weyl (hence sixteen-component) fermion. Both A_μ and ψ are promoted to the $N \times N$ hermitian matrices. This model incorporates the $SO(9,1)$ Lorentz symmetry and the $SU(N)$ gauge symmetry. This model is a totally reduced model in that it is reduced to zero dimension. The prominent feature of this model is that it has no free parameter. The overall coefficient g can be trivially absorbed into the fields by the following redefinition.

$$A_\mu \to g^{\frac{1}{2}} A_\mu, \;\; \psi \to g^{\frac{3}{4}} \psi. \tag{2.2}$$

The coefficient g is an artifact of the coupling constant of the super Yang-Mills theory before being reduced, and is nothing but a scaling parameter.

The path integral of the IIB matrix model is given by

$$Z = \int dA d\psi e^{-S_{(E)}}, \tag{2.3}$$

where $S_{(E)}$ is defined in the ten-dimensional Euclidean space by the Wick rotation of A_0 and Γ^0 in the action (2.1). The convergence of the path integral of the IIB matrix model is not trivial, because the gauge group $SU(N)$ is not compact. However, the convergence of the path integral is discussed in the paper [24, 28] (the review is found in the Ph.D. thesis of Austing [34]). The authors corroborated that the path integral converges for[1] $d = 4, 6, 10$. Only for the bosonic part, the path integral is shown to converge for $d \geq d_c$. Here, $d_c = 5$ for the $SU(2)$ gauge group, $d_c = 4$ for the $SU(3)$ and $d_c = 3$ for all the other simple Lie group.

[1] d is the dimension of the IIB matrix model, and the model (2.1) is defined for $d = 10$. More explicitly, the d-dimensional

2.1.1 Relation to the type IIB superstring theory

The IIB matrix model has a deep relation to the type IIB superstring. Here, we discern that the IIB matrix model is obtained by the matrix regularization of the Green-Schwarz action of the type IIB superstring theory. The Green-Schwarz formalism is defined so that the spacetime supersymmetry should be more manifest. The bosonic Nambu-Goto action is given by

$$S_b = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-h} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu. \tag{2.4}$$

Here, $h_{\alpha\beta}$ is a metric on the worldsheet, and has a Lorentz signature $\eta^{\alpha\beta} = \text{diag}(-,+)$. A naive guess for a supersymmetric extension is to replace $\partial_\alpha X_\mu$ with

$$\Pi^\mu_\alpha = \partial_\alpha X^\mu - i(\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1 - \bar{\theta}^2\Gamma^\mu\partial_\alpha\theta^2). \tag{2.5}$$

Π^μ_α is trivially invariant under the supersymmetry transformation

$$\delta_S\theta^A = \epsilon^A, \quad \delta_S X^\mu = i(\bar{\epsilon}^1\Gamma^\mu\theta^1 - \bar{\epsilon}^2\Gamma^\mu\theta^2). \tag{2.6}$$

Here, A runs over $1, 2$, and this is an index for the supersymmetry. However, the naive replacement of $\partial_\alpha X_\mu$ with Π^μ_α for the action (2.4) is not enough, because it does not have a κ symmetry and hence it has twice as many degrees of freedom as it should. Instead, we give the following action:

$$S_{GS} = \frac{-1}{4\pi\alpha'}\int d^2\sigma \left(\sqrt{-h}h^{\alpha\beta}\Pi^\mu_\alpha\Pi_{\beta\mu} + 2i\epsilon^{\alpha\beta}\partial_\alpha X^\mu(\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1 + \bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2) + 2\epsilon^{\alpha\beta}(\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1)(\bar{\theta}^2\Gamma_\mu\partial_\alpha\theta^2)\right).$$

In this section, we define the rank-2 epsilon tensor as $\epsilon^{01} = 1$ (and hence $\epsilon_{01} = -1$). This action is defined for the $d = 10$ dimensional spacetime, and the index μ runs over $0, 1, 2, \cdots, 9$. The supersymmetry transformation of this action is

$$\delta_S S_{GS} = \frac{-1}{2\pi\alpha'}\int d^2\sigma\epsilon^{\alpha\beta}[(\bar{\epsilon}^1\Gamma^\mu\partial_\alpha\theta^1)(\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1) - (\bar{\epsilon}^2\Gamma^\mu\partial_\alpha\theta^2)(\bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2)], \tag{2.7}$$

where we drop the surface terms. The term (2.7) is shown to vanish (i.e. the action S_{GS} is supersymmetry invariant) by noting the following rewriting:

$$A = \epsilon^{\alpha\beta}\bar{\epsilon}\Gamma^\mu\partial_\alpha\theta\bar{\theta}\Gamma_\mu\partial_\beta\theta = \bar{\epsilon}\Gamma^\mu\dot{\theta}\bar{\theta}\Gamma_\mu\theta' - \bar{\epsilon}\Gamma^\mu\theta'\bar{\theta}\Gamma_\mu\dot{\theta}. \tag{2.8}$$

Here, $\sigma_0 = \tau$ and $\sigma_1 = \sigma$. $\dot{\theta} = \frac{\partial\theta}{\partial\tau}$ and $\theta' = \frac{\partial\theta}{\partial\sigma}$. A can be further written as

$$A = \frac{2}{3}[\bar{\epsilon}\Gamma^\mu\dot{\theta}\bar{\theta}\Gamma_\mu\theta' + \bar{\epsilon}\Gamma^\mu\theta'\dot{\bar{\theta}}\Gamma_\mu\theta + \bar{\epsilon}\Gamma^\mu\theta\bar{\theta}'\Gamma_\mu\dot{\theta}] + \frac{1}{3}\frac{\partial}{\partial\tau}[\bar{\epsilon}\Gamma^\mu\theta\bar{\theta}\Gamma_\mu\theta'] - \frac{1}{3}\frac{\partial}{\partial\sigma}[\bar{\epsilon}\Gamma^\mu\theta\bar{\theta}\Gamma_\mu\dot{\theta}]. \tag{2.9}$$

The first term vanishes due to the Fierz identity

$$\bar{\epsilon}\Gamma^\mu\psi_{[1}\bar{\psi}_2\Gamma_\mu\psi_{3]} = 0. \tag{2.10}$$

We delegate the proof of this Fierz identity to Appendix A.1.6. The second and third terms vanish because they are nothing but surface terms.

This model also incorporates the symmetry named the κ symmetry:

$$\begin{aligned}\delta_\kappa\theta^A &= 2\Gamma^\mu\Pi^\alpha_\mu\kappa^{A\alpha}, \quad \delta_\kappa X^\mu = i\bar{\theta}^1\Gamma^\mu\delta_\kappa\theta^1 - i\bar{\theta}^2\Gamma^\mu\delta_\kappa\theta^\mu, \\ \delta_\kappa(\sqrt{-h}h^{\alpha\beta}) &= -16i\sqrt{-h}(\partial_\gamma\bar{\theta}^1\kappa^{1\beta}P^{\alpha\gamma}_- - \partial_\gamma\bar{\theta}^2\kappa^{2\beta}P^{\alpha\gamma}_+).\end{aligned} \tag{2.11}$$

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supersymmetric IIB matrix model means the following action

$$S = -\frac{1}{g^2}Tr\left(\frac{1}{4}\sum_{\mu,\nu=0}^{d-1}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\sum_{\mu=0}^{d-1}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right),$$

which is well-defined for $d = 3, 4, 6, 10$. The explicit definition of the d-dimensional bosonic IIB matrix model is given by

$$S = -\frac{1}{g^2}Tr\left(\frac{1}{4}\sum_{\mu,\nu=0}^{d-1}[A_\mu, A_\nu][A^\mu, A^\nu]\right).$$

Here, $P_{\pm}^{\alpha\beta}$ is a projection operator defined by

$$P_{\pm}^{\alpha\beta} = \frac{1}{2}(h^{\alpha\beta} \pm \frac{\epsilon^{\alpha\beta}}{\sqrt{-h}}). \tag{2.12}$$

This satisfies the following properties

$$P_{\pm}^{\alpha\beta}h_{\beta\gamma}P_{\pm}^{\gamma\delta} = P_{\pm}^{\alpha\delta}, \quad P_{\pm}^{\alpha\beta}h_{\beta\gamma}P_{\mp}^{\gamma\delta} = 0, \quad P_{\pm}^{\alpha\beta}P_{\pm}^{\gamma\delta} = P_{\pm}^{\gamma\beta}P_{\pm}^{\alpha\delta}, \tag{2.13}$$

which follow from the inversion rule

$$h^{\alpha\beta} = \frac{1}{h}\begin{pmatrix} h_{11} & -h_{01} \\ -h_{10} & h_{00} \end{pmatrix}. \tag{2.14}$$

Here, the parameters of the κ-symmetry is subject to the following projection rule:

$$P_{-}^{\alpha\beta}\kappa_{\beta}^{1} = \kappa^{1\alpha}, \quad P_{+}^{\alpha\beta}\kappa_{\beta}^{2} = \kappa^{2\alpha}. \tag{2.15}$$

The κ-symmetry transformation varies the action as

$$\begin{aligned} \delta_{\kappa}S_{GS} \;=\; & \frac{-1}{2\pi\alpha'}\int d^2\sigma\epsilon^{\alpha\beta}\left([(\partial_{\beta}\bar{\theta}^1)\Gamma^{\mu}(\partial_{\alpha}\theta^1)][\bar{\theta}^1\Gamma_{\mu}\delta_{\kappa}\theta^1] + 2[(\partial_{\alpha}\bar{\theta}^1)\Gamma^{\mu}\delta_{\kappa}\theta^1][\bar{\theta}^1\Gamma_{\mu}\partial_{\beta}\theta^1]\right. \\ & \left. -[(\partial_{\beta}\bar{\theta}^2)\Gamma^{\mu}(\partial_{\alpha}\theta^2)][\bar{\theta}^2\Gamma_{\mu}\delta_{\kappa}\theta^2] - 2[(\partial_{\alpha}\bar{\theta}^2)\Gamma^{\mu}\delta_{\kappa}\theta^2][\bar{\theta}^2\Gamma_{\mu}\partial_{\beta}\theta^2]\right), \end{aligned} \tag{2.16}$$

up to the surface terms that vanish in the integral. We can show that the Green-Schwarz action is invariant under the κ-symmetry with the help of the identity (2.10).

Nextly, we explain how the matrix regularization of the Green-Schwarz action of the type IIB superstring theory leads to the IIB matrix model. For the type IIB superstring theory, the chirality of the two spinors θ^1 and θ^2 is identical. Therefore, we set $\theta^1 = \theta^2 = \theta$. This simplifies the term Π_{α}^{μ} as $\Pi_{\alpha}^{\mu} = \partial_{\alpha}X^{\mu}$. Then, the action is simplified as

$$S_{GS} = \frac{-1}{4\pi\alpha'}\int d^2\sigma\left(h^{\alpha\beta}\sqrt{-h}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu} + 4i\epsilon^{\alpha\beta}\partial_{\alpha}X^{\mu}\bar{\theta}\Gamma_{\mu}\partial_{\beta}\theta\right). \tag{2.17}$$

Integrating out $h_{\alpha\beta}$, we obtain

$$\begin{aligned} S_{GS} &= \frac{-1}{2\pi\alpha'}\int d^2\sigma\left(\sqrt{-\det(\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu})} + 2i\epsilon^{\alpha\beta}\partial_{\alpha}X^{\mu}\bar{\theta}\Gamma_{\mu}\partial_{\beta}\theta\right) \\ &= \frac{-1}{2\pi\alpha'}\int d^2\sigma\left(\sqrt{-\frac{1}{2}(\epsilon^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu})^2} + 2i\epsilon^{\alpha\beta}\partial_{\alpha}X^{\mu}\bar{\theta}\Gamma_{\mu}\partial_{\beta}\theta\right). \end{aligned} \tag{2.18}$$

In the last equality, we note that this determinant can be rewritten as

$$\begin{aligned} \det(\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu}) &= (\partial_0X^{\mu}\partial_0X_{\mu})(\partial_1X^{\nu}\partial_1X_{\nu}) - (\partial_0X^{\mu}\partial_1X_{\mu})(\partial_1X^{\nu}\partial_0X_{\nu}) \\ &= \frac{1}{2}(\epsilon^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu})^2. \end{aligned} \tag{2.19}$$

We introduce the Schild action and discern that this reduces to the Green-Schwarz action. The Schild action is defined by

$$S_{\text{Schild}} = -\int d^2\sigma\left(\sqrt{-h_s}a\left(-\frac{1}{4N}\{X_{\mu}, X_{\nu}\}^2 + \frac{i}{2}\bar{\psi}\Gamma^{\mu}\{X_{\mu}, \psi\}\right) + \sqrt{-h_s}bN\right). \tag{2.20}$$

Here, we introduce the scalar density $\sqrt{-h_s}$ as an independent variable. The Poisson bracket $\{q, p\}$ is defined by

$$\{q, p\} = \frac{1}{\sqrt{-h_s}}\epsilon^{\alpha\beta}\partial_{\alpha}q\partial_{\beta}p. \tag{2.21}$$

Varying this Schild action with respect to $\sqrt{-h_s}$, we obtain

$$\sqrt{-h_s} = \frac{1}{2N}\sqrt{\frac{a}{b}}\sqrt{-(\epsilon^{\alpha\beta}\partial_{\alpha}X^{\mu}\partial_{\beta}X_{\mu})^2}. \tag{2.22}$$

This scalar density is clearly identical to the determinant in the Green-Schwarz action $\sqrt{-h}$ up to a constant. Plugging this into the action, we obtain

$$S_{\text{Schild}} = -\int d^2\sigma \left(\sqrt{ab}\sqrt{-(\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu)^2} + \frac{ia}{2}\epsilon^{\alpha\beta}\bar{\psi}\Gamma^\mu \partial_\alpha X_\mu \partial_\beta \psi \right), \tag{2.23}$$

which reduces to the Green-Schwarz action by setting $(a,b) = (\frac{2}{\pi\alpha'}, \frac{1}{16\pi\alpha'})$. We discuss the matrix regularization of the Schild action (2.20). We perform the following replacement:

$$-i\{,\} = N[,], \quad \int d^2\sigma = \frac{1}{N}Tr. \tag{2.24}$$

By setting $a = \frac{1}{g^2}$ and dropping the term $\sqrt{-h_s}b$, the Schild action arrives at the IIB matrix model.

We next discuss the supersymmetry of the Green-Schwarz action. We identify the two combinations of the parameter of the original supersymmetry and the κ symmetry, to obtain a new supersymmetry transformation. To this end, we relate the parameters of the supersymmetry and the κ symmetry as

$$\kappa^{1\alpha} = -\frac{1}{4}\Gamma^\mu(\epsilon^1 - \epsilon^2)\Pi_{\beta\mu}P_-^{\alpha\beta}, \quad \kappa^{2\alpha} = \frac{1}{4}\Gamma^\mu(\epsilon^1 - \epsilon^2)\Pi_{\beta\mu}P_+^{\alpha\beta}. \tag{2.25}$$

We take the following gauge for the Schild action of the type IIB superstring theory:

$$\theta^1 = \theta^2 (= \theta), \quad \Pi^\mu_\alpha = \partial_\alpha X^\mu, \quad \frac{1}{\sqrt{-h_s}}\epsilon^{\alpha\beta}\Pi^\mu_\alpha \Pi^\nu_\beta = \{X^\mu, X^\nu\}. \tag{2.26}$$

Thus, the fermionic supersymmetry transformation is written as

$$\delta_{\kappa^1}\theta^1 = \frac{1}{2}\left(1 - \frac{1}{2}\Gamma^{\mu\nu}\{X_\mu, X_\nu\}\right)(\epsilon^1 - \epsilon^2), \quad \delta_{\kappa^2}\theta^2 = \frac{1}{2}\left(1 + \frac{1}{2}\Gamma^{\mu\nu}\{X_\mu, X_\nu\}\right)(\epsilon^1 - \epsilon^2). \tag{2.27}$$

Thus, the κ symmetry transformation for X^μ is

$$\delta_\kappa X^\mu = -i\bar{\theta}\Gamma^\mu(\epsilon_1 - \epsilon_2). \tag{2.28}$$

We introduce the following redefinition of the supersymmetry parameter:

$$\chi = 2(\epsilon^1 - \epsilon^2), \quad \chi' = \frac{(\epsilon^1 + \epsilon^2)}{2}. \tag{2.29}$$

The combination of the supersymmetry and the κ symmetry is thus given by

$$(\delta_\epsilon + \delta_\kappa)X^\mu = i\bar{\chi}\Gamma^\mu\theta, \quad (\delta_\epsilon + \delta_\kappa)\theta = \chi' + \frac{1}{8}\Gamma^{\mu\nu}\{X_\mu, X_\nu\}\chi. \tag{2.30}$$

2.1.2 Supersymmetry

Now, let us have a careful look at the supersymmetry of the IIB matrix model. We have formulated the supersymmetry of the Schild action of the type IIB superstring. In the IIB matrix model we consider that the matrix-regularized version of that supersymmetry is inherited. The $\mathcal{N} = 2$ supersymmetry of the IIB matrix model is then distinguished by

- homogeneous: $\delta^{(1)}_\epsilon \psi = \frac{i}{2}[A_\mu, A_\nu]\Gamma^{\mu\nu}\epsilon$, $\delta^{(1)}_\epsilon A_\mu = i\bar{\epsilon}\Gamma_\mu\psi$.
  The feature of the *'homogeneous supersymmetry'* is that this supersymmetry transformation depends on the matter fields A_μ and ψ. And this supersymmetry transformation vanishes if there is no matter field.
- inhomogeneous: $\delta^{(2)}_\xi \psi = \xi$, $\delta^{(2)}_\xi A_\mu = 0$.
  The feature of the *'inhomogeneous supersymmetry'* is that the translation survives without the matter fields.

The commutators of these supersymmetries give the following important results:

$$(1) \quad [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]A_\mu = 0, \quad [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]\psi = 0, \tag{2.31}$$

$$(2) \quad [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]A_\mu = 0, \quad [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}] = 0, \tag{2.32}$$

$$(3) \quad [\delta^{(1)}_\epsilon, \delta^{(2)}_\xi]A_\mu = -i\bar{\epsilon}\Gamma_\mu\xi, \quad [\delta^{(1)}_\epsilon, \delta^{(2)}_\xi]\psi = 0. \tag{2.33}$$

These properties can be verified by taking the difference of the two supersymmetry transformations.

1. This is the most complicated to compute. For the gauge field, we should consider the following transformation

$$A_\mu \overset{\delta^{(1)}_{\epsilon_2}}{\to} A_\mu + i\epsilon_2\Gamma_\mu\psi \overset{\delta^{(1)}_{\epsilon_1}}{\to} A_\mu + i(\bar\epsilon_1+\bar\epsilon_2)\Gamma_\mu\psi - \frac{1}{2}\bar\epsilon_2\Gamma_\mu[A_\nu,A_\rho]\Gamma^{\nu\rho}\epsilon_1, \tag{2.34}$$

$$A_\mu \overset{\delta^{(1)}_{\epsilon_1}}{\to} A_\mu + i\epsilon_1\Gamma_\mu\psi \overset{\delta^{(1)}_{\epsilon_2}}{\to} A_\mu + i(\bar\epsilon_1+\bar\epsilon_2)\Gamma_\mu\psi - \frac{1}{2}\bar\epsilon_1\Gamma_\mu[A_\nu,A_\rho]\Gamma^{\nu\rho}\epsilon_2. \tag{2.35}$$

Then, the commutator is

$$[\delta^{(1)}_{\epsilon_1},\delta^{(1)}_{\epsilon_2}]A_\mu = -\frac{1}{2}\bar\epsilon_2\Gamma_\mu[A_\nu,A_\rho]\Gamma^{\nu\rho}\epsilon_1 + \frac{1}{2}\bar\epsilon_1\Gamma_\mu[A_\nu,A_\rho]\Gamma^{\nu\rho}\epsilon_2. \tag{2.36}$$

Utilizing the formula $\Gamma^\mu\Gamma^{\nu\rho} = \Gamma^{\mu\nu\rho} + \eta^{\mu\nu}\Gamma^\rho - \eta^{\mu\rho}\Gamma^\nu$, we obtain

$$[\delta^{(1)}_{\epsilon_1},\delta^{(1)}_{\epsilon_2}]A_\mu = 2\bar\epsilon_1\Gamma^\rho\epsilon_2[A_\mu,A_\rho]. \tag{2.37}$$

For the fermions, we have only to repeat the similar procedure:

$$\psi \overset{\delta^{(1)}_{\epsilon_2}}{\to} \psi + \frac{i}{2}[A_\mu,A_\nu]\Gamma^{\mu\nu}\epsilon_2 \overset{\delta^{(1)}_{\epsilon_1}}{\to} \psi + \frac{i}{2}[A_\mu,A_\nu]\Gamma^{\mu\nu}(\epsilon_1+\epsilon_2) - [A_\mu,\bar\epsilon_1\Gamma_\nu\psi]\Gamma^{\mu\nu}\epsilon_2, \tag{2.38}$$

$$\psi \overset{\delta^{(1)}_{\epsilon_1}}{\to} \psi + \frac{i}{2}[A_\mu,A_\nu]\Gamma^{\mu\nu}\epsilon_1 \overset{\delta^{(1)}_{\epsilon_2}}{\to} \psi + \frac{i}{2}[A_\mu,A_\nu]\Gamma^{\mu\nu}(\epsilon_1+\epsilon_2) - [A_\mu,\bar\epsilon_2\Gamma_\nu\psi]\Gamma^{\mu\nu}\epsilon_1. \tag{2.39}$$

Using the formula of the Fierz transformation,

$$\bar\epsilon_1\Gamma_\nu\psi\Gamma^{\mu\nu}\epsilon_2 = \bar\epsilon_1\Gamma^\mu\epsilon_2\psi - \frac{7}{16}\bar\epsilon_1\Gamma^\rho\epsilon_2\Gamma_\rho\Gamma^\mu\psi - \frac{1}{16\times 5!}\bar\epsilon_1\Gamma^{\rho_1\cdots\rho_5}\epsilon_2\Gamma_{\rho_1\cdots\rho_5}\Gamma^\mu\psi + (\text{rank 3 term}), \tag{2.40}$$

whose proof we present in Appendix A.1.5, we verify that the commutator of the supersymmetry transformation is

$$[\delta^{(1)}_{\epsilon_1},\delta^{(1)}_{\epsilon_2}]\psi = 2[\psi,\bar\epsilon_1\Gamma^\rho\epsilon_2 A_\rho]. \tag{2.41}$$

Here, we utilize the equation of motion

$$\frac{dS}{d\bar\psi} = -\frac{1}{g^2}\Gamma^\mu[A_\mu,\psi] = 0, \tag{2.42}$$

in order to eliminate the second and the third terms of the Fierz identity (2.40).

Next, we note that the commutators of the supersymmetry transformation (2.37) and (2.41) vanish up to the gauge transformation. The gauge transformation of the IIB model is to multiply the unitary matrix $i\alpha \in SU(N)$. The gauge transformation is expressed in the infinitesimal form as follows :

$$A_\mu,\psi \to A_\mu + i[A_\mu,\alpha], \quad \psi + i[\psi,\alpha]. \tag{2.43}$$

The supersymmetry transformation (2.37) and (2.41) can be gauged away by the gauge parameter $\alpha = 2\bar\epsilon_1\Gamma^\rho\epsilon_2 A_\rho$. We now complete the proof of (2.31) up to the gauge transformation.

2. This is trivial because the supersymmetry $\delta^{(2)}_\xi$ involves only a constant.

3. This can be proven by taking the difference of these two transformations

$$\begin{aligned} A_\mu &\overset{\delta^{(2)}_\xi}{\to} A_\mu \overset{\delta^{(1)}_\epsilon}{\to} A_\mu + i\bar\epsilon\Gamma_\mu\psi, \text{ whereas } A_\mu \overset{\delta^{(1)}_\epsilon}{\to} A_\mu + i\bar\epsilon\Gamma_\mu\psi \overset{\delta^{(2)}_\xi}{\to} A_\mu + i\bar\epsilon\Gamma_\mu(\psi+\xi), \\ \psi &\overset{\delta^{(2)}_\xi}{\to} \psi+\xi \overset{\delta^{(1)}_\epsilon}{\to} \psi+\xi+\frac{i}{2}\Gamma^{\mu\nu}[A_\mu,A_\nu]\epsilon, \text{ whereas} \\ \psi &\overset{\delta^{(1)}_\epsilon}{\to} \psi + \frac{i}{2}\Gamma^{\mu\nu}[A_\mu,A_\nu]\epsilon \overset{\delta^{(2)}_\xi}{\to} \psi+\xi+\frac{i}{2}\Gamma^{\mu\nu}[A_\mu,A_\nu]\epsilon. \end{aligned} \tag{2.44}$$

This completes the proof of the commutation relation of the supersymmetry transformation. When we take the linear combination of the homogeneous and the inhomogeneous supersymmetry as

$$\tilde{\delta}^{(1)}_\epsilon = \delta^{(1)}_\epsilon + \delta^{(2)}_\epsilon, \ \ \tilde{\delta}^{(2)}_\epsilon = i(\delta^{(1)}_\epsilon - \delta^{(2)}_\epsilon), \tag{2.45}$$

the commutator is written as

$$[\tilde{\delta}^{(\alpha)}_\epsilon, \tilde{\delta}^{(\beta)}_\xi]\psi = 0, \ \ [\tilde{\delta}^{(\alpha)}_\epsilon, \tilde{\delta}^{(\beta)}_\xi]A_\mu = -2i\bar{\epsilon}\Gamma_\mu\xi\delta^{\alpha\beta}, \tag{2.46}$$

where α, β run over $1, 2$. The commutation relation (2.46) has a serious consequence. When we regard the eigenvalues of the bosonic matrices A_μ as the spacetime coordinate, the IIB matrix model carries a spacetime $\mathcal{N} = 2$ supersymmetry. Namely, the commutator of the supersymmetry transformations gives a translation of the spacetime by $a_\mu = -2i\bar{\epsilon}\Gamma_\mu\xi$. This urges us to interpret the spacetime as emerging from the eigenvalues of A_μ. This is a crucial property in order for this model to include the gravitational interaction. If the IIB matrix model has a massless particle, it must have a spin-2 graviton. We have a good reason to reduce not the four or six-dimensional, but the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory. This model has the 32 maximal supersymmetry only if we reduce the ten-dimensional super Yang-Mills theory. The maximal supersymmetry is an essential aspect for the theory of gravity.

2.2 Interaction between the BPS object

We next review the interpretation of the BPS objects (such as the D-brane) in terms of the IIB matrix model. We start with considering the classical equation of motion of the IIB matrix model:

$$[A_\nu, [A_\mu, A_\nu]] = 0. \tag{2.47}$$

Here, we set the fermion ψ to zero. The corresponding equation of motion for the type IIB superstring theory is given by

$$\{X_\nu, \{X^\mu, X^\nu\}\} = 0. \tag{2.48}$$

In terms of the type IIB superstring theory, this has a solution representing the D1-brane:

$$X_0 = T\tau, \ \ X_1 = \frac{L}{2\pi}\sigma, \ \ X_2 = X_3 = \cdots = X_9 = 0, \tag{2.49}$$

where T and L are the compactification radii of X^0 and X^1, respectively. The parameter τ and σ take values $0 \leq \tau \leq 1$ and $0 \leq \sigma \leq 2\pi$. The Poisson bracket is given by

$$\{X_0, X_1\} = \epsilon^{01}\partial_0 X_0 \partial_1 X_1 = T \times \frac{L}{2\pi} = \frac{TL}{2\pi}. \tag{2.50}$$

The corresponding commutator for the matrix regularization is

$$-i[A_0, A_1] = \frac{TL}{2\pi N} \Leftrightarrow \{X_0, X_1\} = \frac{TL}{2\pi}. \tag{2.51}$$

This commutation relation is realized only for the infinite-size matrices (we can see that this is impossible for the finite-size matrices by using the cyclic rule of the trace Tr). Namely, we have the following solution:

$$A_0 = \frac{T}{\sqrt{2\pi N}}q, \ \ A_1 = \frac{L}{\sqrt{2\pi N}}p, \ \ A_2 = A_3 = \cdots = A_9 = 0. \tag{2.52}$$

where q, p are the infinite-size canonical pair satisfying $[q, p] = i$.

We discern that the solution (2.52) is a BPS saturated state in the following way. We substitute the solution (2.52), along with $A_2 = A_3 = \cdots = A_9 = 0$ and $\psi = 0$, into the supersymmetry transformation in the previous section to obtain

$$\begin{aligned}
&\delta^{(1)}_\epsilon\psi = -\frac{TL}{4\pi N}\Gamma^{01}\epsilon, \ \ \delta^{(1)}_\epsilon A_\mu = 0, \\
&\delta^{(2)}_\xi\psi = \xi, \ \ \delta^{(2)}_\xi A_\mu = 0.
\end{aligned} \tag{2.53}$$

The homogeneous supersymmetry can be clearly canceled by the inhomogeneous supersymmetry, when we properly choose the parameter ξ. In this sense, the D1-brane solution of the IIB matrix model (2.52) actually preserves half of the supersymmetry, and thus is a BPS saturated object.

By the same token, we can establish the solutions of the IIB matrix model that correspond to the $D3, D5, \cdots$ brane, by taking the similar canonical pairs as

$$[X_2, X_3] \propto \mathbf{1}_{N\times N}, \;\; [X_4, X_5] \propto \mathbf{1}_{N\times N}, \cdots. \tag{2.54}$$

This nicely corresponds to the fact that the type IIB superstring theory accepts the D1,3,5,$\cdots$-branes.

We next see that the bosonic matrices A_μ of the IIB matrix model describe the multi-body system. We can easily build the solution for the two D1-branes located at $x_2 = \pm\frac{b}{2}$ by taking the following block-diagonal matrices.

$$\begin{aligned} & A_0 = \begin{pmatrix} \frac{T}{\sqrt{2\pi n_1}} q & 0 \\ 0 & \frac{T}{\sqrt{2\pi n_2}} q' \end{pmatrix}, \;\; A_1 = \begin{pmatrix} \frac{T}{\sqrt{2\pi n_1}} p & 0 \\ 0 & \frac{T}{\sqrt{2\pi n_2}} p' \end{pmatrix}, \;\; A_2 = \begin{pmatrix} \frac{b}{2} & 0 \\ 0 & -\frac{b}{2} \end{pmatrix}, \\ & A_3 = \cdots = A_9 = 0, \end{aligned} \tag{2.55}$$

where $n_{1,2}$ is the size of each block, and q, q', p, p' satisfy $[q,p] = [q',p'] = i$.

By the same token, we can build the $D1 - \bar{D}1$ brane system through the following classical solution.

$$\begin{aligned} & A_0 = \begin{pmatrix} \frac{T}{\sqrt{2\pi n_1}} q & 0 \\ 0 & \frac{T}{\sqrt{2\pi n_2}} q' \end{pmatrix}, \;\; A_1 = \begin{pmatrix} \frac{T}{\sqrt{2\pi n_1}} p & 0 \\ 0 & -\frac{T}{\sqrt{2\pi n_2}} p' \end{pmatrix}, \;\; A_2 = \begin{pmatrix} \frac{b}{2} & 0 \\ 0 & -\frac{b}{2} \end{pmatrix}, \\ & A_3 = \cdots = A_9 = 0, \end{aligned} \tag{2.56}$$

In this way, the IIB matrix model describes not only the single D-objects , but also the multi-body

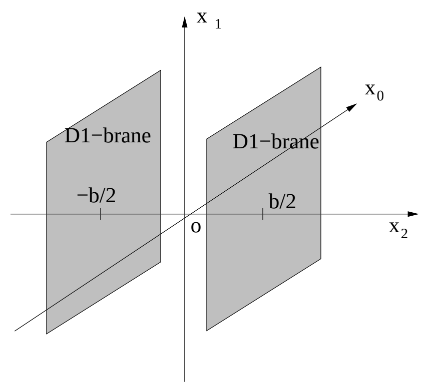

Figure 1: The two parallel D1-branes, represented by the classical solution of the IIB matrix model (2.55).

system by allocating the plural matters on the block-diagonal components. In this sense, there is a great difference from the action of the D-instanton action, which appears to be the same action. Unlike the D-instanton action, the IIB matrix model accepts the multi-body system and thus is a second-quantized action.

Not only do the matrices of the IIB matrix model A_μ accommodate the multi-body system but also the interaction is embedded in the same matrices. Namely, the off-diagonal part can be interpreted as representing the interaction of each block.

We next investigate the interaction between the D-objects by the one-loop calculation. To this end, we derive the one-loop effective action. We separate the matrices with the classical solution and the fluctuation as

$$A_\mu = p_\mu + a_\mu, \;\; \psi = \chi + \varphi. \tag{2.57}$$

Here, p_μ and χ are the classical solution, and a_μ and φ are the fluctuation. Since the IIB matrix model is invariant under the $SU(N)$ gauge transformation (2.43), we need the gauge fixing. Here, we adopt the background gauge

$$-i[p_\mu, A^\mu] = 0. \tag{2.58}$$

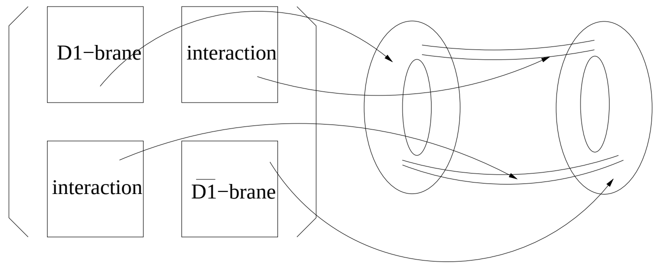

Figure 2: The off-diagonal parts represent the interaction in the IIB matrix model.

The corresponding gauge-fixing term $S_{\text{g.f.}}$ and the ghost term S_{ghost} are respectively

$$S_{\text{g.f.}} = -\frac{1}{2}Tr[p_\mu, A^\mu]^2, \quad S_{\text{ghost}} = -Tr([p_\mu, b][p^\mu, c]), \tag{2.59}$$

where c and b are the ghosts and the anti-ghosts respectively. In the following, we set the parameter g to 1. Then, we extract the quadratic term of the fluctuation for the total action $S^{(\text{total})} = S + S_{\text{g.f.}} + S_{\text{ghost}}$ as

$$\begin{aligned} S_2^{(\text{total})} &= -Tr\left(\frac{1}{2}[p_\mu, a_\nu]^2 + [p_\mu, p_\nu][a^\mu, a^\nu] + \frac{1}{2}\bar{\varphi}\Gamma^\mu[p_\mu, \varphi] + [p_\mu, b][p^\mu, c]\right) \\ &= Tr\left(\frac{1}{2}a^\mu(P^2\delta_{\mu\nu} - 2iF_{\mu\nu})a^\nu + \frac{1}{2}\bar{\varphi}\Gamma^\mu P_\mu\varphi - bP^2c\right). \end{aligned} \tag{2.60}$$

Here, we introduce the following adjoint operators

$$P_\mu X = (\text{ad}p_\mu)X = [p_\mu, X], \quad F_{\mu\nu}X = (\text{ad}(-i[p_\mu, p_\nu]))X = -i[P_\mu, P_\nu]X, \tag{2.61}$$

where the last equality for $F_{\mu\nu}$ follows from the Jacobi identity of the commutators. Integrating the fluctuation, we obtain the following effective action.

$$\begin{aligned} W &= -\log\int dadbdcd\varphi \exp(-S_2^{(\text{total})}) \\ &= \frac{1}{2}Tr\left(tr_{10\times10}\log(P^2\delta_{\mu\nu} - 2iF_{\mu\nu}) - \log P^2 - \frac{1}{4}tr_{16\times16}\log\left(P^2 + \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\frac{1+\Gamma^\sharp}{2}\right)\right) \end{aligned} \tag{2.62}$$

Here, the trace Tr is for the $N \times N$ matrices, while the trace $tr_{10\times10}$ and $tr_{16\times16}$ are respectively for the ten-dimensional vector indices and the ten-dimensional gamma matrices ($\frac{1+\Gamma^\sharp}{2}$ is the Weyl projector, and effectively the size is not 32 but 16).

We evaluate the interaction between the diagonal blocks using the one-loop effective action (2.62), and elucidate that the IIB matrix model includes the gravitational interaction. We consider the backgrounds having a block-diagonal form:

$$A_\mu = p_\mu = \text{diag}(p_\mu^{(1)}, p_\mu^{(2)}, \cdots). \tag{2.63}$$

Here, each block $p_\mu^{(i)}$ represents the separate D-object and is an $n_i \times n_i$ matrix. We decompose the classical background between the trace and the traceless part as

$$p_\mu^{(i)} = d_\mu^{(i)} 1_{n_i\times n_i} + \tilde{p}_\mu^{(i)}. \tag{2.64}$$

Since the eigenvalues of the bosonic matrices are regarded as the spacetime coordinate in the IIB matrix model, the trace $d_\mu^{(i)}$ should be identified with the collective coordinate of the D-objects. We assume that the classical D-objects are separated with each other so that the distances $d_\mu^{(i)} - d_\mu^{(j)}$ are sufficiently large.

The adjoint operator P_μ operates on the (i,j) components of the matrix $X^{(i,j)}$ as

$$(P_\mu X)^{(i,j)} = (d_\mu^{(i)} - d_\mu^{(j)})X^{(i,j)} + \tilde{p}_\mu^{(i)}X^{(i,j)} - X^{(i,j)}\tilde{p}_\mu^{(j)} = ((d_\mu^{(i)} - d_\mu^{(j)}) + P_{L,\mu}^{(i,j)} + P_{R,\mu}^{(i,j)})X^{(i,j)}, \tag{2.65}$$

where $P^{(i,j)}_{L,\mu}X^{(i,j)} = \tilde{p}^{(i)}_\mu X^{(i,j)}$ and $P^{(i,j)}_{R,\mu} = -X^{(i,j)}\tilde{p}^{(j)}_\mu$ represent the operation of P_μ from the left and right, respectively. Similarly, we simplify the notation of the operation of the adjoint operator $F_{\mu\nu}$. The background field strength is rewritten as

$$f_{\mu\nu} = \mathrm{diag}(i[p^{(1)}_\mu, p^{(1)}_\nu], i[p^{(2)}_\mu, p^{(2)}_\nu], \cdots) = \mathrm{diag}(\tilde{f}^{(1)}_{\mu\nu}, \tilde{f}^{(2)}_{\mu\nu}, \cdots). \tag{2.66}$$

The operation of $F_{\mu\nu}$ can be likewise decomposed into the left and right operation as

$$(F_{\mu\nu}X)^{(i,j)} = \tilde{f}^{(i)}_{\mu\nu}X^{(i,j)} - X^{(i,j)}\tilde{f}^{(j)}_{\mu\nu} = (F^{(i,j)}_{L,\mu\nu} + F^{(i,j)}_{R,\mu\nu})X^{(i,j)}. \tag{2.67}$$

Since the left and right operation are independent, we have

$$TrO = \sum_{i,j=1}^{n} TrO^{(i,j)}_L TrO^{(i,j)}_R. \tag{2.68}$$

We obtain the one-loop interaction for these D-objects. To this end, we expand the effective action (2.62) with respect to the power of F_μ. The first bosonic term is expanded as

$$\begin{aligned}
&\frac{1}{2}Tr\left(tr_{10\times 10}\log(P^2\delta_{\mu\nu} - 2iF_{\mu\nu})\right) \\
= \;& Tr\left(5\log P^2 + \frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\nu\mu} - \frac{4i}{3}\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\nu\rho}\frac{1}{P^2}F_{\rho\mu} - 2\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\nu\rho}\frac{1}{P^2}F_{\rho\chi}\frac{1}{P^2}F_{\chi\mu}\right) + \mathcal{O}(P^{-10}).
\end{aligned}$$

Whereas, the third fermionic term of (2.62) is expanded as

$$\begin{aligned}
&-\frac{1}{4}Tr\left(tr_{16\times 16}\log\left(P^2 + \frac{i}{2}F_{\mu\nu}\Gamma^{\mu\nu}\frac{1+\Gamma^\sharp}{2}\right)\right) \\
= \;& Tr\left(-4\log P^2 - \frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\nu\mu} + \frac{4i}{3}\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\nu\rho}\frac{1}{P^2}F_{\rho\mu} + \frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\nu\rho}\frac{1}{P^2}F_{\rho\chi}\frac{1}{P^2}F_{\chi\mu}\right. \\
& + \frac{1}{4}\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\rho\chi}\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\rho\chi} + \frac{1}{2}\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\rho\chi}\frac{1}{P^2}F_{\rho\chi} \\
& \left. -2\frac{1}{P^2}F_{\mu\nu}\frac{1}{P^2}F_{\mu\rho}\frac{1}{P^2}F_{\chi\nu}\frac{1}{P^2}F_{\chi\rho}\right) + \mathcal{O}(P^{-10}).
\end{aligned}$$

In this derivation, we utilize the following formulae of the gamma matrices:

$$\begin{aligned}
& tr(\Gamma^{\mu_1\mu_2}\Gamma^{\nu_1\nu_2}) = -2tr(\eta^{[\mu_1[\nu_1}\eta^{\mu_2]\nu_2]}) \\
& tr(\Gamma^{\mu_1\mu_2}\Gamma^{\nu_1\nu_2}\Gamma^{\rho_1\rho_2}) = 8tr(\eta^{[\mu_1[\nu_1}\eta^{\mu_2][\rho_1}\eta^{\nu_2]\rho_2]}), \\
& tr(\Gamma^{\mu_1\mu_2}\Gamma^{\nu_1\nu_2}\Gamma^{\rho_1\rho_2}\Gamma^{\chi_1\chi_2}) = tr(4\eta^{[\mu_1[\nu_1}\eta^{\mu_2]\nu_2]}\eta^{[\rho_1[\chi_1}\eta^{\rho_2]\chi_2]} + 8\eta^{[\mu_1[\rho_1}\eta^{\mu_2]\rho_2]}\eta^{[\nu_1[\chi_1}\eta^{\nu_2]\chi_2]} \\
& \qquad -16\eta^{[\mu_1[\nu_1}\eta^{[\rho_1[\chi_1}\eta^{\mu_2]\chi_2]}\eta^{\nu_2]\chi_2]} - 16\eta^{[\mu_1[\rho_1}\eta^{\mu_2][\chi_1}\eta^{[\nu_1\chi_2]}\chi^{\nu_2]\chi_2]} \\
& \qquad +16\eta^{[\mu_1[\nu_1}\eta^{[\rho_1[\chi_1}\chi^{\mu_2]\chi_2]}\eta^{\nu_2]\rho_2]}).
\end{aligned}$$

Along with the second ghost term $-Tr\log P^2$, we discern that the effect of the boson and the fermion cancels up to the $\mathcal{O}(P^{-6})$ order. This cancellation is ascribed to the supersymmetry of the IIB matrix model. Due to this cancellation, the leading effect of the one-loop interaction is of the order $\mathcal{O}(P^{-8})$. Namely, the interaction complies with the power law $\sim \frac{1}{r^{d-2}}$ (where r is the distance of the two D-objects, and the dimensionality is $d = 10$). In this sense, we regard this one-loop interaction as the gravitational interaction. This is an important evidence that the IIB matrix model describes the gravitational interaction, and this is a consequence of the supersymmetry. Especially, the contribution of the (i,j) block is expressed using the notation for the left and right operation as

$$\begin{aligned}
W^{(i,j)} \;=\;& \frac{n_j}{4(d^{(i)}-d^{(j)})^8}Tr\left(-4(\tilde{f}^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\nu\rho}\tilde{f}^{(i)}_{\rho\chi}\tilde{f}^{(i)}_{\chi\mu}) - 8(\tilde{f}^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\rho\chi}\tilde{f}^{(i)}_{\mu\chi}\tilde{f}^{(i)}_{\rho\nu}) + 2(\tilde{f}^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\rho\chi}\tilde{f}^{(i)}_{\rho\chi}) + (\tilde{f}^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\rho\chi}\tilde{f}^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\rho\chi})\right) \\
+\;& \frac{n_i}{4(d^{(i)}-d^{(j)})^8}Tr\left(-4(\tilde{f}^{(j)}_{\mu\nu}\tilde{f}^{(j)}_{\nu\rho}\tilde{f}^{(j)}_{\rho\chi}\tilde{f}^{(j)}_{\chi\mu}) - 8(\tilde{f}^{(j)}_{\mu\nu}\tilde{f}^{(j)}_{\rho\chi}\tilde{f}^{(j)}_{\mu\chi}\tilde{f}^{(j)}_{\rho\nu}) + 2(\tilde{f}^{(j)}_{\mu\nu}\tilde{f}^{(j)}_{\mu\nu}\tilde{f}^{(j)}_{\rho\chi}\tilde{f}^{(j)}_{\rho\chi}) + (\tilde{f}^{(j)}_{\mu\nu}\tilde{f}^{(j)}_{\rho\chi}\tilde{f}^{(j)}_{\mu\nu}\tilde{f}^{(j)}_{\rho\chi})\right) \\
+\;& \frac{1}{4(d^{(i)}-d^{(j)})^8}\left(-48Tr(\tilde{f}^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\nu\rho})Tr(\tilde{f}^{(j)}_{\mu\chi}\tilde{f}^{(j)}_{\chi\rho}) + 6Tr(\tilde{f}^{(i)}_{\mu\nu}\tilde{f}^{(i)}_{\mu\nu})Tr(\tilde{f}^{(j)}_{\rho\chi}\tilde{f}^{(j)}_{\rho\chi})\right) + \mathcal{O}((d^{(i)}-d^{(j)})^{-10}).
\end{aligned} \tag{2.69}$$

The tensor structure indicates that the first and second term of the last line of (2.69) represent the graviton and dilaton exchange respectively. This result augurs very well for the IIB matrix model to be a bona fide framework for the gravitation interaction. This argument is limited to the background gauge, and it is an interesting future work to extend this argument in a gauge independent way.

2.3 Interpretation of the diffeomorphism invariance

In this subsection, we review the interpretation of the diffeomorphism invariance on the target space for the IIB matrix model. In [14], it has been pointed out that the invariance under the permutation of the eigenvalues S_N is interpreted as the diffeomorphism invariance of the low-energy action. Since the permutation invariance is of course a subgroup of the $SU(N)$ symmetry, this means that both the diffeomorphism invariance and the gauge invariance emerge from the $SU(N)$ invariance! This is a surprising aspect of the IIB matrix model, in that the unification of these two symmetries has been achieved in a natural way.

In discussing the dynamics of the spacetime, we expand the IIB matrix model around the diagonal background

$$p_\mu = \mathrm{diag}(x^1_\mu, x^2_\mu, \cdots, x^N_\mu), \quad \chi = \mathrm{diag}(\chi^1, \chi^2, \cdots, \chi^N), \tag{2.70}$$

where these satisfy the constraint $\sum_{i=1}^N x^i_\mu = 0$ and $\sum_{i=1}^N \chi^i = 0$, because they belong to the $SU(N)$ gauge group. We delegate the details to the references [11, 14, 16, 75], but after we integrate the fluctuation at the one-loop level and further the fermion zero mode χ, we finally obtain the following interesting result:

$$\int dX d\chi \exp(-S^{\text{one-loop}}_{\text{eff}}[x, \chi]) = \sum_{G:\text{graph}} \int dX W[X; G]. \tag{2.71}$$

Here, G denotes the graphs that connects the eigenvalues X, and W is the Boltzmann weight for the graph G and the configuration of the eigenvalues X. An explicit calculation also indicates that the dependence on the two different eigenvalues x^i and x^j contributes at the order $(x^i - x^j)^{-12}$, when x^i and x^j are connected by the graph G.

The important property of the integral (2.71) is that it is invariant under the permutation of the eigenvalues S_N. We note that this is of course not the case with each graph G, because the way to connect the eigenvalues by the graphs impairs the S_N invariance. Nevertheless, this symmetry is retrieved when we take a summation for all the graphs. This phenomenon is somewhat reminiscent of the dynamical triangulation approach to the quantum gravity, in which the diffeomorphism invariance is retrieved by summing over all the triangulation.

Here, we see how the permutation invariance of the eigenvalues leads to the diffeomorphism invariance. In a sense, this is a very natural interpretation, since we identify the eigenvalues with the spacetime coordinate. To elaborate on this viewpoint, we consider the scalar field ϕ^i propagating in the distributed eigenvalues. Namely, we consider the following effective action as an example;

$$S = \sum_{i,j} \frac{(\phi^i - \phi^j)^2}{2} f(x^i - x^j) + \sum_i m(\phi^i)^2. \tag{2.72}$$

$f(x)$ is a function damping fast sufficiently at infinity. We introduce the density function $\rho(x) = \sum_i \delta^{(10)}(x - x^i)$. Along with the continuous function $\phi(x)$ that satisfies $\phi^i = \phi(x^i)$, the action (2.72) is rewritten as

$$S = \int dx dy \langle \rho(x)\rho(y) \rangle \frac{(\phi(x) - \phi(y))^2}{2} f(x - y) + m \int dx \langle \rho(x) \rangle \phi(x)^2. \tag{2.73}$$

Here, we take an average with respect to the eigenvalue configuration X and the graphs G. In order to see clearly the correspondence between the gravitational background, we further rewrite the action as

$$\begin{aligned} S &= \frac{1}{2} \int dx \langle \rho(x) \rangle \left[\int dy \langle \rho(y) \rangle (x - y)_\mu (x - y)_\nu f(x - y)(1 + c(x, y)) \right] \partial^\mu \phi(x) \partial^\nu \phi(x) \\ &+ m \int dx \langle \rho(x) \rangle \phi^2(x) + \cdots . \end{aligned} \tag{2.74}$$

Here, we normalize the density correlation as $\langle \rho(x)\rho(y) \rangle = \langle \rho(x) \rangle \langle \rho(y) \rangle (1 + c(x, y))$. This urges us to identify the eigenvalue density with the gravitational background as

$$\sqrt{g} e^{-\Phi(x)} \sim \langle \rho(x) \rangle, \quad g_{\mu\nu}(x) = \int dy \langle \rho(y) \rangle (x - y)_\mu (x - y)_\nu f(x - y)(1 + c(x, y)). \tag{2.75}$$

Nextly, we see how the diffeomorphism invariance is realized. Since the IIB matrix model itself is $SU(N)$ invariant, the model itself is of course invariant under its subgroup S_N. Under the permutation $x^i \to x^{\sigma(i)}$ for $\sigma \in S_N$, the fields ϕ^i transform to $\phi^{\sigma(i)}$. We extend the permutation of the eigenvalue to the transformation of the continuous transformation $x \to \xi(x)$, where $\xi(x)$ is a continuous function such that $\xi(x^i) = x^{\sigma(i)}$. It is trivial that $\phi(x)$ is subject to the general coordinate transformation as a scalar field. In addition, the background metric given in (2.75) also receives a transformation as a rank-2 tensor, when the function $f(x)$ decreases rapidly and has a support only near the origin $x = 0$. In this sense, we can fairly interpret the S_N invariance as the general coordinate invariance of the low-energy limit.

While the IIB matrix model has only a flat noncommutative background and depends heavily on the flat metric, we have an interpretation for the curved space background. A nontrivial background is induced dynamically through the condensation of the graviton.

2.4 Alternative totally reduced models

In this subsection, we review the attempts for the generalization of the IIB matrix model. The IIB matrix model augurs well for the unification of the gravity. The $\mathcal{N} = 2$ supersymmetry is a fundamental object for the theory of gravity. In the preceding two sections, we have reviewed some of the evidences for the gravitational interaction: the graviton-dilaton exchange in the one-loop calculation and the interpretation of the diffeomorphism invariance. However, the IIB matrix model suffers a serious drawback in that it has only the flat noncommutative background. However, if a matrix model is to be a bona fide framework to describe the gravitational interaction, it should accommodate a curved space background in a natural way. In this section, we introduce several generalizations of the IIB matrix model, so that they should incorporate a curved space background.

2.4.1 The IIB matrix model with the Chern-Simons term

Firstly, we introduce the matrix model with the Chern-Simons term, defined by the following action [25]:

$$S = \frac{1}{g^2} Tr \left(-\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{2i\alpha}{3}\epsilon_{\mu\nu\rho} A^\mu A^\nu A^\rho + \frac{1}{2}\bar{\psi}\sigma^\mu [A_\mu, \psi] \right). \tag{2.76}$$

Here, the indices $\mu, \nu, \rho, \cdots$ run over $1, 2, 3$ and this model is defined on the three-dimensional Euclidean space. A_μ are the three-dimensional bosonic vectors and ψ is the three-dimensional Majorana fermion. This is also a totally reduced model like the IIB matrix model, and A_μ and ψ are promoted to the $N \times N$ hermitian matrices. σ_μ are the Pauli matrices and in the following $\sigma_{\mu\nu}$ denotes $\sigma_{\mu\nu} = \frac{1}{2}[\sigma_\mu, \sigma_\nu] = i\epsilon_{\mu\nu\rho}\sigma_\rho$. This model has the $SO(3)$ rotational symmetry and the $SU(N)$ gauge symmetry.

In [56], Tomino performed the calculation of the path integral for the $N = 2$ case as a toy model, and elucidated that the path integral converges for the $N = 2$ case (while the bosonic model (5.1) which we analyze in Sec. 5 does diverge). This is in contrast to the IIB matrix model without the Chern-Simons term, in which the path integral diverges for $d = 3$ and arbitrary N. Austing and Wheater in [57] corroborated that adding the Chern-Simons term does not affect the convergence as long as the original path integral (without the Chern-Simons term) converges absolutely.

The interesting property of this matrix model is that it incorporates the classical solution of the fuzzy sphere. Its classical equation of motion is given by

$$[A_\mu, [A_\mu, A_\nu]] + i\alpha\epsilon_{\nu\rho\chi}[A_\rho, A_\chi] = 0. \tag{2.77}$$

The fuzzy-sphere solution is given by the N-dimensional irreducible representation of the $SU(2)$ Lie algebra L_μ:

$$A_\mu = \alpha L_\mu, \text{ where } [L_\mu, L_\nu] = i\epsilon_{\mu\nu\rho}L_\rho. \tag{2.78}$$

This is regarded as the sphere, firstly because it satisfies the relation

$$A_1^2 + A_2^2 + A_3^2 = R^2 \mathbf{1}_{N\times N} = \alpha^2 \frac{N^2-1}{4}\mathbf{1}_{N\times N}. \tag{2.79}$$

The relation (2.79) determines the radius of the sphere as $R^2 = \alpha^2\frac{N^2-1}{4}$. In addition, the eigenvalues of A_μ are distributed sphere-like. In this sense, this classical solution is interpreted as the sphere in the

three-dimensional spacetime. That this matrix model has the S^2 fuzzy sphere classical solution is a great advantage. While the S^2 fuzzy sphere is nothing but a simple manifold, it is of significance as a prototype of the curved space background, which is a fundamental principle of the general relativity.

This matrix model also incorporates the $\mathcal{N}=2$ supersymmetry. Like the IIB matrix model, this model also has the homogeneous and the inhomogeneous supersymmetry.

$$\text{homogeneous: } \delta^{(1)}_\epsilon A_\mu = i\bar{\epsilon}\sigma_\mu\psi, \;\; \delta^{(1)}_\epsilon \psi = \frac{i}{2}([A_\mu, A_\nu] - i\alpha\epsilon_{\mu\nu\rho}A_\rho)\sigma^{\mu\nu}\epsilon, \tag{2.80}$$

$$\text{inhomogeneous: } \delta^{(2)}_\epsilon A_\mu = 0, \;\; \delta^{(2)}_\epsilon \psi = \xi. \tag{2.81}$$

This model has the similar commutation relation to that of the IIB matrix model:

$$[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]A_\mu = 0, \;\; [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]\psi = 0, \tag{2.82}$$

$$[\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]A_\mu = 0, \;\; [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]\psi = 0, \tag{2.83}$$

$$[\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]A_\mu = -i\bar{\epsilon}\sigma_\mu\xi, \;\; [\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]\psi = 0. \tag{2.84}$$

The proof of these commutation relations goes in the same way as in the IIB matrix model, and we give only the proof of (2.82). Using the formula $\sigma_\mu\sigma_{\nu\rho} = \delta_{\mu\nu}\sigma_\rho - \delta_{\mu\rho}\sigma_\nu$, we obtain the commutation relation

$$[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]A_\mu = i[A_\mu, \lambda] + \epsilon_{\mu\nu\rho}\theta_\nu A_\rho, \tag{2.85}$$

where $\lambda = 2i(\bar{\epsilon}_2\sigma_\mu\epsilon_1)A_\mu$ and $\theta_\mu = 2i(\bar{\epsilon}_2\sigma_\mu\epsilon_1)$. For the fermion, we obtain the commutator by taking the difference likewise:

$$[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]\psi = \psi - ([\bar{\epsilon}_1\sigma_\mu\psi, A_\nu]\sigma_{\mu\nu}\epsilon_2 + \alpha(\bar{\epsilon}_1\sigma_\rho\psi)\sigma_\rho\epsilon_2) + (\epsilon_1 \leftrightarrow \epsilon_2). \tag{2.86}$$

We use the following Fierz transformation whose detailed proof we give in Appendix A.1.5:

$$(\bar{\epsilon}_1\sigma_\mu\psi)\sigma_{\mu\nu}\epsilon_2 = -(\bar{\epsilon}_1\sigma_\nu\epsilon_2)\psi + (\text{rank-0 terms}), \;\; (\bar{\epsilon}_1\sigma_\mu\psi)\sigma_\mu\epsilon_2 = \frac{1}{2}(\bar{\epsilon}_1\sigma_\mu\epsilon_2)\sigma_\mu\psi + (\text{rank-0 terms}). \tag{2.87}$$

In (2.87), we omit the rank-0 contribution, since we are interested in the commutator $[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]\psi$. Then, we obtain the commutation relation

$$[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]\psi = i[\psi, \lambda] - i\sigma_\mu\frac{\theta_\mu}{2}\psi. \tag{2.88}$$

The difference from the IIB matrix model is that the commutation relation (2.88) is satisfied without using the classical equation of motion. The relations (2.85) and (2.88) indicate that the commutators vanish up to the $SO(3)$ rotation by θ_μ, as well as the gauge transformation by λ. The appearance of the $SO(3)$ symmetry is a novelty of this matrix model.

We likewise take the linear combination of the supersymmetry as

$$\tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)}, \;\; \tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)}). \tag{2.89}$$

Their commutation relation gives the translation of the bosonic matrices A_μ as

$$[\tilde{\delta}^{(\alpha)}_\epsilon, \tilde{\delta}^{(\beta)}_\xi]\psi = 0, \;\; [\tilde{\delta}^{(\alpha)}_\epsilon, \tilde{\delta}^{(\beta)}_\xi]A_\mu = -2i(\bar{\epsilon}\sigma_\mu\xi)\delta^{\alpha\beta}. \tag{2.90}$$

The fuzzy-sphere classical solution is a BPS saturated state. Plugging the classical solution (2.78) and $\psi = 0$ into the homogeneous transformation (2.80), we immediately find that the homogeneous supersymmetry vanishes. Whereas, the inhomogeneous supersymmetry (2.81) survives. In this sense, the fuzzy-sphere solution (2.78) preserves half of the supersymmetry.

We next discuss the expansion of this matrix model around the fuzzy sphere solution (2.78). In the case of the IIB matrix model, we derive the noncommutative Yang-Mills theory by the expansion around the flat noncommutative background [17]. This derivation of the noncommutative Yang-Mills theory is quite natural, because the matrices, per se, are noncommutative objects. The studies of the noncommutative field theory have become explosively popular since Seiberg and Witten [18] elucidated the relation to the superstring theory. In the noncommutative spacetime, we impose the noncommutativity on the spacetime as $[x_\mu, x_\nu] = ic_{\mu\nu}$. This relationship is reminiscent of the canonical commutation relation of the space and

the momentum in the quantum mechanics. Just as the spacetime and momentum have the uncertainty principle, the noncommutative spacetime incorporates the uncertainty of the spacetime. In the context of the superstring theory, the noncommutativity of the spacetime is introduced by turning on the B field [18], which gives the noncommutativity

$$c_{\mu\nu} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B} \right)_A = -(2\pi\alpha') \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)_{\mu\nu}. \tag{2.91}$$

This leads to the uncertainty of the spacetime at the string scale $\mathcal{O}(\sqrt{\alpha'}) \sim \mathcal{O}(l_s)$.

When we expand the matrix model (2.76) around the fuzzy sphere background (2.78), we obtain a noncommutative Yang-Mills theory on the S^2 sphere. The expansion around the sphere is given by

$$A_\mu = \alpha L_\mu + \alpha R \hat{a}_\mu, \tag{2.92}$$

where $\hat{a}_\mu$ is the fluctuation. The fluctuation is expanded by the noncommutative spherical harmonics as

$$\hat{a}_\mu = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{\mu lm} \hat{Y}_{lm}, \quad \hat{\psi} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} \psi_{lm} \hat{Y}_{lm}, \tag{2.93}$$

where the noncommutative spherical harmonics $\hat{Y}_{lm}$ is given by

$$\hat{Y}_{lm} = R^{-l} \sum_a f^{lm}_{a_1 \cdots a_l} \hat{x}^{a_1} \hat{x}^{a_2} \cdots \hat{x}^{a_l}. \tag{2.94}$$

The indices a_μ run over $1, 2, 3$, and $f^{lm}_{a_1 \cdots a_l}$ is a symmetric and traceless tensor. We can understand that this is a $(2l+1)$-dimensional tensor as follows. Firstly, when we allocate $1, 2, 3$ to $a_1, \cdots, a_l$ in a symmetric way, we have ${}_{l+2}C_2 = \frac{(l+2)(l+1)}{2}$ ways. On the other hand, the tracelessness condition implies that $f^{lm}_{aaa_1a_2\cdots a_{l-2}}$ vanishes. This condition stems from the constraint $(\hat{x}^a)^2 = \mathbf{1}_{N\times N}$. This eliminates ${}_lC_2 = \frac{l(l-1)}{2}$ ways. Therefore, the dimensionality of this tensor is $\frac{(l+2)(l+1)}{2} - \frac{l(l-1)}{2} = 2l+1$. Its normalization is defined by

$$\frac{1}{N} Tr(\hat{Y}_{lm} \hat{Y}^\dagger_{l'm'}) = \delta_{ll'} \delta_{mm'}. \tag{2.95}$$

This of course corresponds to the expansion in terms of the c-number spherical harmonics

$$a_\mu(\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{\mu lm} Y_{lm}(\Omega), \quad \psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_{lm} Y_{lm}(\Omega), \text{ where} \tag{2.96}$$

$$Y_{lm}(\Omega) = R^{-l} \sum_a f^{lm}_{a_1 \cdots a_l} x^{a_1} \cdots x^{a_l}, \quad \int \frac{d\Omega}{4\pi} Y_{lm} Y^*_{l'm'} = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta Y_{lm} Y^*_{l'm'} = \delta_{ll'} \delta_{mm'}. \tag{2.97}$$

Now, we can build the mapping rule between the matrices and the functions on the S^2 sphere. The rule is listed below:

$$\hat{a}_\mu = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{\mu lm} \hat{Y}_{lm} \to a_\mu(\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{\mu lm} Y_{lm}(\Omega), \tag{2.98}$$

$$\frac{1}{N} Tr \to \int \frac{d\Omega}{4\pi}, \tag{2.99}$$

$$\hat{a}\hat{b} \to a(\Omega) \star b(\Omega). \tag{2.100}$$

We need to look more closely at the star product (2.100). Firstly, the correspondence (2.98) is rewritten, using the orthogonality condition (2.95), as

$$\hat{a}_\mu \to a_\mu(\Omega) = \frac{1}{N} \sum_{l=0}^{N-1} \sum_{m=-l}^{l} Tr(\hat{a}_\mu \hat{Y}^\dagger_{lm}) Y_{lm}(\Omega). \tag{2.101}$$

Here, we require that the maximum value of l is $N-1$. This leads us to define the star product as follows. Here, we require that $l+l'$ should not exceed $N-1$.

$$\begin{aligned} a\star b(\Omega) &= \frac{1}{N}\sum_{l=0}^{N-1}\sum_{m=-l}^{l} Tr(\hat{a}\hat{b}\hat{Y}^{\dagger}_{lm})Y_{lm}(\Omega) \\ &= \frac{1}{N}\sum_{lm}\sum_{l'm'}\sum_{l''m''}\int d\Omega' d\Omega'' Y^{*}_{l'm'}(\Omega')Y^{*}_{l''m''}(\Omega)a(\Omega')b(\Omega'')Y_{lm}(\Omega)Tr(\hat{Y}_{l'm'}\hat{Y}_{l''m''}\hat{Y}^{\dagger}_{lm}). \end{aligned} \tag{2.102}$$

Then, the action (2.76) is mapped as follows:

$$\begin{aligned} S &= -\frac{\alpha^4R^4}{4g^2}Tr(\hat{F}_{\mu\nu}\hat{F}_{\mu\nu}) - \frac{\alpha^4}{6g^2}TrL_\mu^2 - \frac{i}{2g^2}\alpha^4R^3\epsilon_{\mu\nu\rho}Tr\left(\frac{1}{R}[L_\mu,\hat{a}_\nu]\hat{a}_\rho + \frac{1}{3}\hat{a}_\mu[\hat{a}_\nu,\hat{a}_\rho] - \frac{i}{2R}\epsilon_{\nu\rho\chi}\hat{a}_\mu\hat{a}_\chi\right) \\ &+ \frac{\alpha}{2g^2}Tr\bar{\hat{\psi}}\sigma^\mu[L_\mu + R\hat{a}_\mu,\psi] \\ &= -\frac{\alpha^4(N^3-N)}{24g^2} - \frac{\alpha^4R^4N}{4g^2}\int\frac{d\Omega}{4\pi}\left\{(F_{\mu\nu}F_{\mu\nu}) + \frac{2i}{R}\epsilon_{\mu\nu\rho}\left(\frac{1}{R}(L_\mu a_\nu)a_\rho + \frac{1}{3}a_\mu[a_\nu,a_\rho] - \frac{i}{2R}\epsilon_{\nu\rho\chi}a_\mu a_\chi\right)\right\}_\star \\ &+ \frac{\alpha RN}{2g^2}\int\frac{d\Omega}{4\pi}\left(\frac{1}{R}\bar{\psi}\sigma^\mu L_\mu\psi + \bar{\psi}\sigma^\mu[a_\mu,\psi]\right)_\star. \end{aligned} \tag{2.103}$$

where we define the field strength $\hat{F}_{\mu\nu}$ as

$$\hat{F}_{\mu\nu} = \frac{1}{R}([L_\mu,a_\nu] - [L_\nu,a_\mu]) - \frac{i}{R}\epsilon_{\mu\nu\rho}a_\rho + [a_\mu,a_\nu]. \tag{2.104}$$

The subscript $\star$ indicates that the product is defined by the noncommutative star product. Therefore, the commutator survives even for the $U(1)$ abelian gauge group of the Yang-Mills theory.

This gives the relationship between the coupling constant of the Yang-Mills theory g_{YM} and the *artifact* of the coupling constant of the Yang-Mill theory before being reduced as

$$g_{YM}^2 = \frac{4\pi g^2}{\alpha^4R^2N}. \tag{2.105}$$

We go on to the symmetry of this Yang-Mills theory. Firstly, the $SU(N)$ symmetry is interpreted as the gauge symmetry. For the element of the $SU(N)$ Lie *group* U, it is expanded as $U = \exp(i\lambda) \sim 1 + i\lambda + \cdots$, where λ is an element of the $su(N)$ Lie *algebra*. The gauge transformation for A_μ is given by

$$\delta A_\mu = [A_\mu, i\lambda] = [\alpha L_\mu + \alpha R\hat{a}_\mu, i\lambda]. \tag{2.106}$$

This transforms the fluctuation as

$$\delta\hat{a}_\mu = \frac{i}{R}\alpha[L_\mu,\lambda] + i[\hat{a}_\mu,\lambda]. \tag{2.107}$$

This is immediately translated in terms of the Yang-Mills theory as

$$\delta a_\mu = \frac{i}{R}L_\mu\lambda + i[\hat{a}_\mu,\lambda]_\star. \tag{2.108}$$

Secondly, we note that this noncommutative Yang-Mills theory preserves the following $\mathcal{N}=1$ supersymmetry.

$$\delta_S a_\mu = \frac{i}{\alpha R}\bar{\epsilon}\sigma_\mu\psi, \quad \delta_S\psi = \frac{i\alpha^2R^2}{2}F_{\mu\nu}\sigma^{\mu\nu}\epsilon. \tag{2.109}$$

We have hitherto discussed the perturbation around the irreducible representation of the fuzzy sphere (2.78). In this case, we obtain a $U(1)$ abelian noncommutative Yang-Mills theory. The matrix model (2.76) also accommodates the fuzzy sphere solution of the reducible representation

$$A_\mu = \alpha(L_\mu\otimes\mathbf{1}_{k\times k}) = \begin{pmatrix} \alpha L_\mu & & & \\ & \alpha L_\mu & & \\ & & \ddots & \\ & & & \alpha L_\mu \end{pmatrix}. \tag{2.110}$$

When L_μ is the $n\times n$ representation of the $SU(2)$ Lie algebra, the size of the matrices N satisfies $N = nk$. The perturbation around this reducible representation gives the $U(k)$ nonabelian theory.

2.4.2 Other attempts for higher-dimensional fuzzy sphere solution

We next review the extension of the above discussion to the higher-dimensional fuzzy sphere classical solution. To this end, there are two strategies. First is the addition of the higher-dimensional Chern-Simons term, and second is the addition of the tachyonic mass term.

The first attempt is achieved by defining the following action [40, 50] :

$$S = -\frac{1}{g^2}\left(\frac{1}{4}Tr[A_\mu, A_\nu]^2 + \frac{\lambda}{2k+1}\epsilon_{\mu_1\cdots\mu_{2k+1}} Tr A_{\mu_1}A_{\mu_2}\cdots A_{\mu_{2k+1}}\right). \tag{2.111}$$

Here, we define this matrix model in the $d = (2k+1)$-dimensional Euclidean space. The indices $\mu, \nu, \cdots$ run over $1, 2, \cdots, (2k+1)$. The supersymmetric Yang-Mills theory can be defined only for the $d = 3, 4, 6, 10$ case, and the supersymmetric matrix model cannot be defined except for the three-dimensional case as we have reviewed before. Hence, we focus on the bosonic matrix model in the following. Except for the $k = 1$ (three-dimensional) case, the convergence of the path integral is not yet manifestly proven. Nevertheless, we surmise that the fuzzy-sphere classical solution might be a metastable state, in that it is barriered by a high potential. We attribute this metastability to the similar reasoning to the ϕ^4 matrix model. As we explain in Appendix C, the ϕ^4 matrix model retains the metastability near the origin despite its unbounded-below potential.

The illuminating feature of this higher-dimensional matrix model is that it incorporates the classical solution of the S^{2k} fuzzy sphere. The equation of motion is given by

$$[A_\nu, [A_\mu, A_\nu]] + \lambda\epsilon_{\mu\nu_1\cdots\nu_{2k}} A_{\nu_1}A_{\nu_2}\cdots A_{\nu_{2k}} = 0. \tag{2.112}$$

It has the following classical solution of the S^{2k} fuzzy sphere [10, 31, 43]:

$$\begin{aligned} A_\mu &= \frac{\alpha}{2}G^{(2k)}_\mu, \text{ where} \\ G^{(2k)}_\mu &= (\Gamma^{(2k)}_\mu \otimes \mathbf{1}_{2^k\times 2^k} \otimes \cdots \otimes \mathbf{1}_{2^k\times 2^k})\text{sym} + \cdots + (\mathbf{1}_{2^k\times 2^k} \otimes \mathbf{1}_{2^k\times 2^k} \otimes \cdots \otimes \Gamma^{(2k)}_\mu)\text{sym}. \end{aligned} \tag{2.113}$$

We define the symmetric tensor product more explicitly, for $k = 1, 2$ case (namely, for the S^2 and S^4 fuzzy sphere). We denote the matrix element using the orthonormal state $|i\rangle$ and $|j\rangle$ as $(A)_{ij} = \langle i|A|j\rangle$. The usual tensor product (two-fold product, for brevity) is defined by

$$\langle i_1, i_2|A_1 \otimes A_2|j_1, j_2\rangle = \langle i_1|A_1|j_1\rangle\langle i_2|A_2|j_2\rangle, \text{ where } |j_1, j_2\rangle = |j_1\rangle|j_2\rangle. \tag{2.114}$$

On the other hand, the symmetric tensor product is defined as

$$\text{sym}\langle i_1, i_2|A \otimes B|j_1 j_2\rangle\text{sym}, \text{ where } |j_1, j_2\rangle\text{sym} = \frac{1}{n}(|j_1\rangle|j_2\rangle + |j_2\rangle|j_1\rangle). \tag{2.115}$$

The coefficient n is determined, so that the symmetrized state may retain the orthonormality. Here, we give a more explicit definition for the two-fold tensor product of the Paulian matrices ($k = 1$ case). For the two-fold tensor product, the symmetrized orthonormal state is given by

$$|1, 1\rangle = |1\rangle|1\rangle, \quad |2, 2\rangle = |2\rangle|2\rangle, \quad |1, 2\rangle = |2, 1\rangle = \frac{1}{\sqrt{2}}(|1\rangle|2\rangle + |2\rangle|1\rangle). \tag{2.116}$$

This difference comes from the fact that the states $|1, 1\rangle$ and $|2, 2\rangle$ are symmetrized ab ovo and hence that we do not need symmetrization or normalization. We elaborate on these definitions in terms of explicit representation in terms of matrices. The usual tensor product (2.114) is rewritten as

$$A \otimes B = \left(\begin{array}{c|cccc} & |j_1=1, j_2=1\rangle & |j_1=2, j_2=1\rangle & |j_1=1, j_2=2\rangle & |j_1=2, j_2=2\rangle \\ \hline \langle i_1=1, i_2=1| & a_{11}b_{11} & a_{12}b_{11} & a_{11}b_{12} & a_{12}b_{12} \\ \langle i_1=2, i_2=1| & a_{21}b_{11} & a_{22}b_{11} & a_{21}b_{12} & a_{22}b_{12} \\ \langle i_1=1, i_2=2| & a_{11}b_{21} & a_{12}b_{21} & a_{11}b_{22} & a_{12}b_{22} \\ \langle i_1=2, i_2=2| & a_{21}b_{21} & a_{22}b_{21} & a_{21}b_{22} & a_{22}b_{22} \end{array}\right). \tag{2.117}$$

On the other hand, its symmetrization is rewritten as

$$(A\otimes B)_{\text{sym}} = \left(\begin{array}{c|ccc} & |1,1\rangle_{\text{sym}} & |1,2\rangle_{\text{sym}} & |2,2\rangle_{\text{sym}} \\ \hline {}_{\text{sym}}\langle 1,1| & a_{11}b_{11} & \frac{(a_{12}b_{11}+a_{11}b_{12})}{\sqrt{2}} & a_{12}b_{12} \\ {}_{\text{sym}}\langle 1,2| & \frac{(a_{21}b_{11}+a_{11}b_{21})}{\sqrt{2}} & \frac{(a_{22}b_{11}+a_{21}b_{12}+a_{12}b_{21}+a_{11}b_{22})}{(\sqrt{2})^2} & \frac{(a_{22}b_{12}+a_{12}b_{22})}{\sqrt{2}} \\ {}_{\text{sym}}\langle 2,2| & a_{21}b_{21} & \frac{(a_{22}b_{21}+a_{21}b_{22})}{\sqrt{2}} & a_{22}b_{22} \end{array}\right). \tag{2.118}$$

The coefficient λ is related to the radius parameter of the fuzzy sphere α as

$$\lambda = (\frac{\alpha}{2})^{3-2k}\frac{8k}{m_k}, \tag{2.119}$$

where the coefficient m_k is given later in the self-dual relation relation (2.126).

The S^{2k} fuzzy sphere is mathematically more involved than the S^2 fuzzy sphere. We introduce the properties of the higher-dimensional fuzzy sphere solution. Firstly, the gamma matrices $\Gamma^{(2k)}_\mu$ is defined by the Wick rotation of those for the Minkowski spacetime given in Appendix A. The definition is given by

$$\begin{aligned} &\Gamma^{(2)}_p = \sigma_p, \quad (p=1,2,3), \quad \Gamma^{(2k)}_p = \Gamma^{(2k-2)}_p \otimes \sigma_3 = \begin{pmatrix} \Gamma^{(2k)}_p & 0 \\ 0 & -\Gamma^{(2k)}_p \end{pmatrix}, \\ &\Gamma^{(2k)}_{2k} = \mathbf{1}_{2^{k-1}\times 2^{k-1}} \otimes \sigma_2 = \begin{pmatrix} 0 & -i\mathbf{1}_{2^{k-1}\times 2^{k-1}} \\ i\mathbf{1}_{2^{k-1}\times 2^{k-1}} & 0 \end{pmatrix}, \\ &\Gamma^{(2k)}_{2k+1} = \mathbf{1}_{2^{k-1}\times 2^{k-1}} \otimes \sigma_1 = \begin{pmatrix} 0 & \mathbf{1}_{2^{k-1}\times 2^{k-1}} \\ \mathbf{1}_{2^{k-1}\times 2^{k-1}} & 0 \end{pmatrix}, \end{aligned} \tag{2.120}$$

where p runs over $p = 1, 2, \cdots, 2k-1$. Namely, we identify $i\Gamma_0$ in the Minkowski space with $\Gamma^{(2k)}_{2k}$ in the Euclidean spacetime.

The size of the representation (2.113) is given in [31] as

$$\begin{aligned} &N_1 = (n+1), \quad N_2 = \frac{(n+1)(n+2)(n+3)}{6}, \quad N_3 = \frac{(n+1)(n+2)(n+3)^2(n+4)(n+5)}{360}, \\ &N_4 = \frac{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)}{302400}. \end{aligned} \tag{2.121}$$

We see a grave difference between the $k=1$ (three-dimensional) case and otherwise. For $k=1$, the representation (2.113) can be defined for an arbitrary integer-size matrices (of course, except for 1). However, this never holds true for $k \geq 2$ case. For example, the representation (2.113) is possible only for a limited integer size; namely $N_2 = 4, 10, 20, \cdots$.

The important property is that $G^{(2k)}_\mu$ does not generically close with respect to the commutator for $k \geq 2$. This is also a tremendous difference between the three-dimensional case and the higher-dimensional case. Here, we denote $G^{(2k)}_{\mu\nu} = [G^{(2k)}_\mu, G^{(2k)}_\nu]$. These satisfy the following relations:

$$G^{(2k)}_\mu G^{(2k)}_\mu = n(n+2k)\mathbf{1}_{N_k\times N_k}, \tag{2.122}$$

$$G^{(2k)}_{\mu\nu} G^{(2k)}_{\mu\nu} = -8kn(n+2k)\mathbf{1}_{N_k\times N_k}, \tag{2.123}$$

$$[G^{(2k)}_{\mu\nu}, G^{(2k)}_\rho] = 4(-\delta_{\mu\rho}G^{(2k)}_\nu + \delta_{\nu\rho}G^{(2k)}_\mu), \tag{2.124}$$

$$[G^{(2k)}_{\mu\nu}, G^{(2k)}_{\rho\chi}] = 4(\delta_{\nu\rho}G^{(2k)}_{\mu\chi} + \delta_{\mu\chi}G^{(2k)}_{\nu\rho} - \delta_{\mu\chi}G^{(2k)}_{\nu\rho} - \delta_{\nu\rho}G^{(2k)}_{\mu\chi}). \tag{2.125}$$

In addition, $G^{(2k)}_\mu$ satisfies the following self-dual relation.

$$\epsilon_{\mu\nu_1\cdots\nu_{2k}} G^{(2k)}_{\nu_1} G^{(2k)}_{\nu_2} \cdots G^{(2k)}_{\nu_{2k}} = m_k G_\mu, \text{ where} \tag{2.126}$$

$$m_1 = 2i, \quad m_2 = 8(n+2), \quad m_k = -2ki(n+2k-2)m_{k-1}. \tag{2.127}$$

Before verifying these properties, we start with convincing ourselves that the notion of the higher-dimensional fuzzy sphere solution is a generalization of the $SU(2)$ Lie algebra. To this end, we focus on the relation between the $k=1$ (three-dimensional) case and the $SU(2)$ Lie algebra.

Firstly, the n-fold symmetric tensor product for $k=1$ is identical to the $(n+1)$-dimensional irreducible representation of the $SU(2)$ Lie algebra. We explicitly give the matrix element of the $n=2$ case as

$$
\begin{aligned}
G_1^{(2)} &= [(\sigma_1 \otimes \mathbf{1}_{2\times 2}) + (\mathbf{1}_{2\times 2} \otimes \sigma_1)]\mathrm{sym} \\
&= \left(\begin{array}{c|cc|c} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \end{array}\right)_{\mathrm{sym}} + \left(\begin{array}{c|cc|c} 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \end{array}\right)_{\mathrm{sym}} = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
G_2^{(2)} &= [(\sigma_2 \otimes \mathbf{1}_{2\times 2}) + (\mathbf{1}_{2\times 2} \otimes \sigma_2)]\mathrm{sym} \\
&= \left(\begin{array}{c|cc|c} 0 & -i & 0 & 0 \\ \hline i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ \hline 0 & 0 & i & 0 \end{array}\right)_{\mathrm{sym}} + \left(\begin{array}{c|cc|c} 0 & 0 & -i & 0 \\ \hline 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ \hline 0 & i & 0 & 0 \end{array}\right)_{\mathrm{sym}} = \sqrt{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \\
G_3^{(2)} &= [(\sigma_3 \otimes \mathbf{1}_{2\times 2}) + (\mathbf{1}_{2\times 2} \otimes \sigma_3)]\mathrm{sym} \\
&= \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right)_{\mathrm{sym}} + \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right)_{\mathrm{sym}} = 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\end{aligned} \tag{2.128}
$$

These are clearly identical to the 3×3 representation of the $SU(2)$ Lie algebra. In addition, the 2-fold tensor product obeys the following important relation:

$$
\begin{aligned}
\sum_{i=1}^{3}(\sigma_\mu \otimes \sigma_\mu)\mathrm{sym} &= \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right)_{\mathrm{sym}} + \left(\begin{array}{c|cc|c} 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 \end{array}\right)_{\mathrm{sym}} + \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)_{\mathrm{sym}} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1}_{3\times 3}.
\end{aligned} \tag{2.129}
$$

Utilizing this relation, the radius of the fuzzy sphere is derived as follows:

$$
\begin{aligned}
(G_\mu^{(2)})^2 &= \underbrace{(\sigma_\mu^2 \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})\mathrm{sym} + \cdots + (\mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \sigma_\mu^2)\mathrm{sym}}_{n \text{ terms}} + \underbrace{(\sigma_\mu \otimes \sigma_\mu \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})\mathrm{sym} + \cdots}_{n(n-1) \text{ terms}} \\
&= (3n + n(n-1))\mathbf{1}_{N_1\times N_1} = (N_1^2 - 1)\mathbf{1}_{N_1\times N_1},
\end{aligned} \tag{2.130}
$$

where we used (2.129) in the second equality and recall that $N_1 = n+1$. This proves the relation (2.122) for $k=1$. This is identical to the Casimir of the J_μ, which satisfies $J_\mu^2 = \frac{N^2-1}{4}$. Here, the relation between $G_\mu^{(2)}$ and J_μ is given by $\frac{1}{2}G_\mu^{(2)} = J_\mu$.

The self-dual relation (2.126) clearly reduces to the commutation relation of the $SU(2)$ Lie algebra:

$$
\epsilon_{\mu\nu\rho}G_\nu^{(2)}G_\rho^{(2)} = \frac{1}{2}\epsilon_{\mu\nu\rho}[G_\nu^{(2)}, G_\rho^{(2)}] = \frac{1}{2}\epsilon_{\mu\nu\rho}(2i\epsilon_{\nu\rho\chi}G_\chi^{(2)}) = 2iG_\mu^{(2)}. \tag{2.131}
$$

We next go into the proof of the relations (2.122) $\sim$ (2.126). We immediately discern that (2.124) and (2.125) are verified from the commutation relations of the gamma matrices. (2.122) can be substantiated by repeating the same argument as for (2.130) in the $k=1$ case. In promoting to the higher-dimensional case, we note that the similar relation to (2.129) holds true of the general case; namely

$$
\sum_{\mu=1}^{2k+1}(\Gamma_\mu^{(2k)} \otimes \Gamma_\mu^{(2k)})\mathrm{sym} = \mathbf{1}. \tag{2.132}
$$

For $k=2$, this relation can be verified as

$$
\sum_{\mu=1}^{5}(\Gamma_\mu^{(2k)} \otimes \Gamma_\mu^{(2k)})\mathrm{sym}
$$

$$= \left(\sum_{\mu=1}^{3}((\sigma_\mu \otimes \sigma_3) \otimes (\sigma_\mu \otimes \sigma_3))_{\rm sym}\right) + ((\mathbf{1} \otimes \sigma_2) \otimes (\mathbf{1} \otimes \sigma_2))_{\rm sym} + ((\mathbf{1} \otimes \sigma_1) \otimes (\mathbf{1} \otimes \sigma_1))_{\rm sym} = \mathbf{1},$$

by using (2.129) twice. We can verify the higher-k case recursively in the same way. Once we prove (2.132), the proof of (2.122) goes in the same way as for (2.130).

The relation (2.123) is verified by the following calculation:

$$\begin{aligned} G^{(2k)}_{\mu\nu} G^{(2k)}_{\mu\nu} &= [(2\Gamma^{\mu\nu} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})_{\rm sym} + \cdots]^2 \\ &= \underbrace{(4\Gamma^{\mu\nu}\Gamma^{\mu\nu} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})_{\rm sym}}_{n \text{ terms}} + \underbrace{4((\Gamma^\mu\Gamma^\nu - \delta^{\mu\nu}) \otimes (\Gamma^\mu\Gamma^\nu - \delta^{\mu\nu}) \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1})_{\rm sym} + \cdots}_{n(n-1) \text{ terms}} \\ &= 4(-n2k(2k+1) - n(n-1)2k)\mathbf{1} = -8kn(n+2k)\mathbf{1}. \end{aligned} \tag{2.133}$$

Lastly, we prove the relation (2.126). The brutal-force proof up to $k = 4$ is given in [46]. However, the Gordian knot is cut in one stroke in [50] by noting the following reduction law. We consider the algebra at the north pole of the fuzzy sphere; $G^{(2k)}_{2k+1} \sim n$. Then, $G^{(2k)}_{ab}$ are the generator of the $SO(2k)$ rotation around the north pole, where the indices $a, b, \cdots$ run over $1, 2, \cdots, 2k$. For $p, q, \cdots = 1, 2, \cdots 2k-1$, the elements for the S^{2k} fuzzy sphere can be identified with those of the S^{2k-2} fuzzy sphere;

$$iG^{(2k)}_{p2k} \sim G^{(2k-2)}_{p}, \quad G^{(2k)}_{pq} \sim G^{(2k-2)}_{pq}. \tag{2.134}$$

This gives the recursive relation of the coefficient m_k as

$$\begin{aligned} m_k G^{(2k)}_{2k+1} &= nm_k = \epsilon_{\mu_1\cdots\mu_{2k}(2k+1)} G^{(2k)}_{\mu_1} \cdots G^{(2k)}_{\mu_{2k}} = 2k\epsilon_{\mu_1\cdots\mu_{2k-1}(2k)(2k+1)} G^{(2k)}_{\mu_1\mu_2} \cdots G^{(2k)}_{\mu_{2k-1}2k} \\ &= -2ki\epsilon_{p_1\cdots p_{2k-1}} G^{(2k-2)}_{p_1} \cdots G^{(2k-2)}_{p_{2k-1}} = -2kim_{k-1}(G^{(2k-2)}_{p_{2k-1}})^2 = -2kim_{k-1}n(n+2k-2). \end{aligned} \tag{2.135}$$

This shows that the coefficient m_k obeys the recursion relation (2.127).

This completes the relations (2.122) $\sim$ (2.126). We can see that (2.113) represents the sphere simply due to the relation (2.122). This gives the radius of the S^{2k} fuzzy sphere as

$$R^2 = \frac{\alpha^2 n(n+2k)}{4}. \tag{2.136}$$

The relation (2.119) has a serious consequence for the radius. For the $k = 1$ case, the radius is proportional to the coefficient λ. However, for $k \geq 2$ case, the radius becomes smaller for the larger coefficient λ.

Nextly, we point out that the S^{2k-2} fuzzy sphere is attached to the S^{2k} fuzzy sphere at each point. This property is understood by considering the algebra on the north pole $G^{(2k)}_{2k+1} \sim n$. As we mentioned before, $G^{(2k)}_{ab}$ are the generators of the $SO(2k)$ rotation. Since identification (2.134) gives the algebra of the S^{2k-2} sphere, we see that the S^{2k-2} sphere is attached on the north pole. While we see only the north pole, it is clear that this holds true of all the points of the S^{2k} sphere due to the rotational symmetry. Due to the identification (2.134), the radius of the S^{2k-2} fuzzy sphere attached to the S^{2k} sphere is clearly given by

$$R^2_{S^{2k-2}} = \frac{\alpha^2 n(n+2k-2)}{4}. \tag{2.137}$$

This property can be understood from the viewpoint of the stabilizer group. The stabilizer group of the S^{2k} fuzzy sphere is $SO(2k+1)/U(k)$. This is identical to

$$\begin{aligned} SO(2k+1)/U(k) &\sim (SO(2k+1)/SO(2k)) \times (SO(2k)/U(k)) \\ &\sim (SO(2k+1)/SO(2k)) \times (SO(2k-1)/U(k-1)). \end{aligned} \tag{2.138}$$

This solidifies the fact that the S^{2k-2} sphere is attached on the S^{2k} sphere.

We see another grave difference between the S^2 and the higher-dimensional S^{2k} fuzzy sphere with respect to the classical stability. The classical energy is easily calculated by plugging the fuzzy sphere solution S^{2k} into the action as

$$S = \frac{kn(n+2k)N_k\alpha^4}{2g^2}\left(\frac{1}{4} - \frac{1}{2k+1}\right). \tag{2.139}$$

In this derivation, we use the relation (2.122) and (2.123), as well as the self-dual relation (2.126). The classical energy (2.139) is negative for the three-dimensional ($k = 1$) case, and thus energetically favored than the flat background $A_\mu = \mathrm{diag}(x_\mu^1, \cdots, x_\mu^N)$, for which $S = 0$. However, the reverse is the case with the $k \geq 2$ case. In this case, the fuzzy sphere solution is *less* energetically favored.

We have seen the alternative matrix model incorporating the higher-dimensional fuzzy sphere solution and the properties of the solution. Another choice is to add the tachyonic mass term to the bosonic part of the IIB matrix model as [29]

$$S = -\frac{1}{g^2} Tr \left(\frac{1}{4}[A_\mu, A_\nu]^2 + \lambda A_\mu^2 \right). \tag{2.140}$$

This model is defined by the general d-dimensional Euclidean space (here, the indices $\mu, \nu, \cdots$ run over $1, 2, \cdots, d$). The convergence of the path integral of this matrix model is not clear, while it is balanced by the mass term and the quartic term. This comes from the diagonal part of the matrices A_μ. The diagonal part of A_μ of course contributes to the tachyonic mass term. Whereas, the quartic term comprises the commutator, so that the diagonal part is canceled. This is a serious drawback, compared with the matrix model with the Chern-Simons term.

The equation of motion

$$[A_\nu, [A_\mu, A_\nu]] + 2\lambda A_\mu = 0 \tag{2.141}$$

gives a variety of the curved-space classical solution. Due to the commutation relation (2.124) and (2.125), (for $d \geq 2k + 1$) it accommodates the S^{2k} fuzzy sphere solution

$$A_\mu = \frac{\alpha}{2} G_\mu, \tag{2.142}$$

where α and the mass λ is now related as

$$\lambda = k\alpha^2. \tag{2.143}$$

The classical energy of the fuzzy sphere solution $A_\mu = \frac{\alpha}{2} G_\mu$ for the action (2.141) is

$$S = -\frac{kn(n + 2k)N_k \alpha^4}{8g^2}. \tag{2.144}$$

This matrix model model has a wider class of the non-trivial classical background than the matrix model with the Chern-Simons term. While we delegate the explicit definition to the paper [29], this matrix model accommodates the classical solution of the fuzzy torus. This property is not shared with the matrix model with the Chern-Simons term. In this sense, the matrix model with a tachyonic mass term is an interesting attempt for a richer curved space background of the matrix model.

2.5 Other related studies

In this brief review, we do not exhaust all the recent developments of the IIB matrix model. Instead, we touch on the other important studies of the IIB matrix model to conclude this review. Firstly, in [11], the reasoning for the IIB matrix model to induce the four-dimensional spacetime dynamically has been found. The novelty is that they related the branched polymer and the eigenvalue distribution of the IIB matrix model. In this way, they found that the Hausdorff dimension of the set of the eigenvalues of the matrices A_μ is four. This is a great discovery, in that the IIB matrix model implies the dynamical generation of our real four-dimensional world. This issue is reviewed in detail in [16, 75].

Another proposal for the dynamical generation of the four-dimensional spacetime is proposed in [36]. In this paper, they discussed the Gaussian expansion of the IIB matrix model, by taking the quadratic action to be the "test action". They conducted the third-order calculation of the Gaussian expansion, and found that the $SO(10)$ Lorentz symmetry breaks down to the $SO(4)$ symmetry. This also serves as an important evidence for the generation of the four-dimensional spacetime. In [39, 48], they refined the calculation by means of the improved Taylor expansion and the notion of the 2PI (two-particle irreducible) Feynman diagram. They substantiated the breakdown of the $SO(10)$ Lorentz symmetry to the $SO(4)$ up to the seventh order of the Gaussian expansion.

The IIB matrix model is a successful proposal for the constructive definition of the superstring theory, and possesses many exciting properties which solidify the confidence that it is an authentic constructive definition.

3 Studies of the $osp(1|32, R)$ supermatrix model

In this section, we discuss the matrix model based on the $osp(1|32, R)$ super Lie algebra, as a natural extension of the IIB matrix model. Firstly, L. Smolin [19] proposed the matrix model based on the $osp(1|32, R)$ super Lie algebra as an M-theory matrix model. Smolin discussed the possibility for this matrix model to induce both the BFSS model [4] and the IIB matrix model [5].

As we have briefly reviewed in the previous section, the IIB matrix model enjoys a lot of successful aspects. And, we have also discussed some of the attempts for the extensions of the IIB matrix model for a more manifest formulation of the gravitational interaction. Here, we introduce the "supermatrix models", which are defined by the "super Lie algebra". There are a lot of illuminating features to the supermatrix model. Firstly, the models include both the bosons and the fermions in a unified way, since the bosons and the fermions are embedded in the same multiplet. Secondly, as we will see more closely in the definition, the supermatrix models accommodate the higher-rank tensor fields, as well as the rank-1 bosonic fields A_μ. Especially, this opens the door to the generalization of the spin connection in terms of the large-N reduced models. If we are to describe a gravitational interaction more manifestly, we should take this question more seriously. Thirdly, the action is described by the cubic term of the supermatrix. This might be a natural aspect for the generalization of the superstring theory, because the cubic interaction is the fundamental one in the sense that the four-point interaction is identical to the connection of the two cubic interactions due to the conformal invariance. Moreover, this model might be solvable in relation to the Chern-Simons theory. The Chern-Simons theory is known to be exactly solvable by means of the Jones Polynomial in the knot theory [73]. If we find a correspondence between the cubic matrix model and Witten's technique of solving the Chern-Simons theory, we may obtain a way to solve the superstring theory exactly in the distant future. This section is based on the author's works [26, 46].

3.1 Definition of the $osp(1|32, R)$ super Lie algebra

Before entering the investigation of the superstring action, we settle the definition of a super Lie algebra. A super Lie algebra is an algebra of supermatrix in which both bosonic matrices and the fermionic matrices are embedded in one matrix. Supermatrices possess many properties different from (ordinary) matrices, and these properties and the notations are summarized in detail in Appendix. A.2.

Let us start with the definition of the $osp(1|32, R)$ super Lie algebra.

- ♣ If $M \in osp(1|32, R)$, then ${}^T MG + GM = 0$ for $G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}$.
- ♣ M is traceless with respect to 33 $\times$ 33 supermatrix.
- ♣ M is a real supermatrix in that $M^* = M$.

We confirm that, for the first condition, $osp(1|32, R)$ forms a closed super Lie algebra. Suppose matrices M_1 and M_2 satisfy the condition

$$ {}^T M_1 G + G M_1 = 0, \quad {}^T M_2 G + G M_2 = 0. \tag{3.1} $$

If $osp(1|32, R)$ is to be a closed super Lie algebra, we call for the following condition:

$$ {}^T([M_1, M_2])G + G[M_1, M_2] = 0. \tag{3.2} $$

Multiplying G^{-1} on both (3.1) and (3.2) from the left, we respectively obtain

$$ G^{-1T} M_k G + M_k = 0, \tag{3.3} $$

$$ G^{-1T}([M_1, M_2])G + [M_1, M_2] = 0, \tag{3.4} $$

where $k = 1, 2$. The proof that $osp(1|32, R)$ is a closed super Lie algebra is equivalent to deriving (3.4) utilizing (3.3):

$$ (3.4) = [G^{-1T} M_2 G, G^{-1T} M_1 G] + [M_1, M_2] \stackrel{(3.3)}{=} [-M_2, -M_1] + [M_1, M_2] = 0. \tag{3.5} $$

This statement, per se, can be satisfied whatever the matrix G may be so long as G has an inverse matrix.

Here comes one question: *Why do we define a metric G as* $G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}$? This stems from the requirement that M should be a real matrix; namely $M^* = ({}^T M)^\dagger = M$. Let us consider the consistency between this reality condition and the very definition of $osp(1|32, R)$ super Lie algebra. Take a hermitian conjugate of the definition ${}^T MG + GM = 0$. This gives

$$G^\dagger ({}^T M)^\dagger + M^\dagger G^\dagger = 0. \tag{3.6}$$

Utilizing the properties introduced in the Appendix. A.2.3, and the reality condition, this is rewritten as

$$0 = G^\dagger ({}^T M)^\dagger + M^\dagger G^\dagger \overset{M^* = ({}^T M)^\dagger}{=} G^\dagger M^* + M^\dagger G^\dagger \overset{M^\dagger = {}^T (M^*)}{=} G^\dagger M^* + {}^T (M^*) G^\dagger \overset{M = M^*}{=} G^\dagger M + {}^T M G^\dagger. \tag{3.7}$$

In order for the relationship (3.7) to be consistent with the very definition of $osp(1|32, R)$, we must call for a condition $G^\dagger = G$. And the starting point of $osp(1|32, R)$ is well-defined if we take a metric as $G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}$.

The next issue is to investigate the explicit form of this super Lie algebra. This is expressed by

$$\text{If } M \in osp(1|32), \text{ then } M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}. \tag{3.8}$$

- m contains only the terms of rank 1,2,5. In other words, m is expressed as

$$m = u_{A_1} \Gamma^{A_1} + \frac{1}{2!} u_{A_1 A_2} \Gamma^{A_1 A_2} + \frac{1}{5!} u_{A_1 \cdots A_5} \Gamma^{A_1 \cdots A_5}. \tag{3.9}$$

- Here $\bar{\psi}$ denotes $\psi^\dagger \Gamma^0$. However, this is equivalent to ${}^T\psi \Gamma^0$, because we are now considering a real super Lie algebra.

The proof of this restriction goes as follows. Let M be the element of $osp(1|32, R)$ and $M = \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix}$:

- $\bar{\phi}$ is defined as $\bar{\phi} = \psi^\dagger \Gamma^0 = {}^T \phi \Gamma^0$.
- Because M is a real supermatrix, m, v and ψ are real, while $i\bar{\phi}$ is pure imaginary (and hence $\bar{\phi}$ is real).

By substituting this formula into the very definition, we obtain

$$\begin{aligned} {}^T MG + GM &= \begin{pmatrix} {}^T m & -i {}^T \bar{\phi} \\ {}^T \psi & {}^T v \end{pmatrix} \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix} + \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix} \\ &= \begin{pmatrix} {}^T m \Gamma^0 + \Gamma^0 m & {}^T \bar{\phi} + \Gamma^0 \psi \\ {}^T \psi \Gamma^0 - \bar{\phi} & 2iv \end{pmatrix} = 0. \end{aligned}$$

We immediately obtain the relationship between two fermionic fields ψ and ϕ from this definition:

$${}^T \bar{\phi} + \Gamma^0 \psi = {}^T \Gamma^0 \phi + \Gamma^0 \psi = -\Gamma^0 \phi + \Gamma^0 \psi = 0. \tag{3.10}$$

By multiplying Γ^0 on the both hand sides from the left , we obtain the relationship between ψ and ϕ. We immediately note that ${}^T \psi \Gamma^0 - \bar{\phi} = 0$ is equivalent to the equation (3.10).

The constraint on v is trivial, and v must vanish. On the other hand, the constraint on m is worth a careful investigation. m is imposed on the constraint

$${}^T m \Gamma^0 + \Gamma^0 m = 0. \tag{3.11}$$

This is the very definition of the Lie algebra called $sp(32)$, and this statement indicates that the bosonic 32×32 matrix of the $osp(1|32, R)$ super Lie algebra must belong to the $sp(32)$ Lie algebra. In analyzing this supermatrix model, it is more convenient to decompose the bosonic part $m \in sp(32)$ in terms of

the basis of arbitrary 32×32 matrices[2]. Namely, we decompose the bosonic part m in terms of the eleven-dimensional gamma matrices $\mathbf{1}_{32\times 32}$, Γ^{A}, $\Gamma^{A_1 A_2}$, $\Gamma^{A_1 A_2 A_3}$, $\Gamma^{A_1\cdots A_4}$ and $\Gamma^{A_1\cdots A_5}$, rather than to obey the expression in the paper [19].

The relationship (3.11) determines what rank of the 11 dimensional gamma matrices survive. Suppose $m \in sp(32)$ are expressed in terms of the gamma matrices:

$$m = u\mathbf{1} + u_A \Gamma^A + \frac{1}{2!} u_{A_1 A_2}\Gamma^{A_1 A_2} + \frac{1}{3!} u_{A_1 A_2 A_3}\Gamma^{A_1 A_2 A_3} + \frac{1}{4!} u_{A_1\cdots A_4}\Gamma^{A_1\cdots A_4} + \frac{1}{5!} u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}. \quad (3.12)$$

The condition (3.11) is rewritten as

$$m = -(\Gamma^0)^{-1}({}^T m)\Gamma^0 = \Gamma^0({}^T m)\Gamma^0. \quad (3.13)$$

Then, performing the following computation for $k = 0, 1, \cdots, 5$, we obtain

$$\Gamma^0({}^T\Gamma^{A_1\cdots A_k})\Gamma^0 = (-1)^{\frac{(k+2)(k-1)}{2}}\Gamma^{A_1\cdots A_k} = \begin{cases} +\Gamma^{A_1\cdots A_k} & (\text{ for } k = 1, 2, 5), \\ -\Gamma^{A_1\cdots A_k} & (\text{ for } k = 0, 3, 4), \end{cases} \quad (3.14)$$

where k is the rank of the gamma matrices. Combining this result with the constraint of the $sp(32)$ Lie algebra (3.13), we discern that only the gamma matrices of rank 1, 2 and 5 survive. We are thus finished with verifying the explicit form of the $osp(1|32, R)$ super Lie algebra.

3.2 $osp(1|32, R)$ (nongauged) cubic matrix model

3.2.1 Action of the cubic supermatrix model

We first discuss the nongauged action of the $osp(1|32, R)$ super Lie algebra (we postpone the meaning of the "(non)gauged action" later). We start with the following action proposed by L. Smolin [19]:

$$\begin{aligned} S &= \frac{i}{g^2} Tr \sum_{Q,R=1}^{33} \left(\left(\sum_{p=1}^{32} M_P^Q [M_Q^R, M_R^P] \right) - M_{33}^Q [M_Q^R, M_R^{33}] \right) \\ &= \frac{i}{g^2} \sum_{a,b,c=1}^{N^2} str_{33\times 33}(M^a M^b M^c) Tr(t^a [t^b, t^c]) = -\frac{f_{abc}}{2g^2} str(M^a M^b M^c). \end{aligned} \quad (3.15)$$

Here, M is a supermatrix belonging to the $osp(1|32, R)$ super Lie algebra. The indices $P, Q, R, \cdots$ and $p, q, r, \cdots$ respectively run over $P, Q, R, \cdots = 1, 2, \cdots, 32, 33$ and $p, q, r, \cdots, = 1, 2, \cdots, 32$. These indices are respectively allocated for the whole 33×33 $osp(1|32, R)$ supermatrix, and the 32×32 bosonic part. Here, we promote each component of the $osp(1|32, R)$ super Lie algebra to the $u(N)$ hermitian matrices. In the following, the lowercase trace and the uppercase trace are for the $32\times 32(33\times 33)$ matrices and the $N \times N$ matrices, respectively. f_{abc} is a structure constant of the $u(N)$ Lie algebra, and i is necessitated in the overall definition of the supermatrix model in order to render the action real.

This model apparently possesses the following pathological properties [19]. The first is that this theory is not bounded from above or below. This pathology stems from the fact that the action is *cubic*, and can be seen by a naive field redefinition $M \to M\lambda$. The action is rewritten as $S \to \lambda^3 S$, and we can set this action to be $S \to \pm\infty$ by setting the parameter to be $\lambda \to \pm\infty$. This means that the path integral of the theory $Z = \int e^{-S}$ does not converge. However, note that this pathology is shared by general relativity. This pathological property of general relativity can be seen by Weyl transformation

$$S_{einstein} = \int d^d x \sqrt{g} R = \int d^d x e^{(\frac{d}{2}-1)\omega} \sqrt{g}(R - (d-1)\nabla^2\omega - \frac{(d-2)(d-1)}{4}\partial_\mu\omega\partial^\mu\omega).$$

Although this pathology of Smolin's proposal is not a happy aspect, this may be regarded as a good news because this may indicate that Smolin's proposals may include general relativity by taking a due limit. The second problem is that this theory possesses no explicit time coordinate. This is again a property shared by the general relativity. We can introduce a time coordinate by expanding the theory around a certain background. Once a time coordinate is introduced, we can construct a Hamiltonian of this theory.

[2]Actually this is a $32 \times 32 = 1024$ dimensional basis, because the dimension of this basis is $1 + {}_{11}C_1 + \cdots {}_{11}C_5 = 1 + 11 + 55 + 165 + 330 + 462 = \frac{1}{2}(1+1)^{11} = 1024$.

We mention the symmetries of this matrix model. Firstly, this model naturally has the $osp(1|32,R)$ symmetry, which is the extension of the $SO(9,1)$ Lorentz symmetry. We note that this symmetry is possible only when we define the theory in terms of the supertrace, which is defined as $strM = \sum_{p=1}^{32} M_p^p - M_{33}$. The cyclic rule of the supermatrix holds true not for the ordinary trace but for the supertrace. This property can be verified by the following argument. We consider the following two arbitrary matrices $Z_1 = \begin{pmatrix} A_{11} & \alpha_{12} \\ \alpha_{21}^T & a_{22} \end{pmatrix}$, $Z_2 = \begin{pmatrix} B_{11} & \beta_{12} \\ \beta_{21}^T & b_{22} \end{pmatrix}$, where A_{11} and B_{11} are 32×32 bosonic matrices, α_{12} , α_{21} , β_{12} and β_{21} are 32-component fermionic vectors, and a_{22} and b_{22} are c-numbers. We consider the supertrace of Z_1Z_2 and Z_2Z_1, which are calculated as $str(Z_1Z_2) = tr(A_{11}B_{11}+\alpha_{12}\beta_{21}^T) - (\alpha_{21}^T\beta_{12}+a_{22}b_{22})$, and $str(Z_2Z_1) = tr(A_{11}B_{11} + \beta_{12}\alpha_{21}^T) - (\beta_{21}^T\alpha_{12} + a_{22}b_{22})$. Since α and β are fermionic quantities, the sign flips if we change the order. Therefore, we establish that it is *not an ordinary trace but a supertrace* that the cyclic rule $str(Z_1Z_2) = str(Z_2Z_1)$ holds true of the supermatrix.

We next discuss its gauge symmetry. This model is invariant under the following $U(N)$ transformation:

$$\delta M = i[(\mathbf{1}_{33\times 33} \otimes u), M], \text{ where } iu \in u(N). \tag{3.16}$$

In this symmetry, we do not mix the $osp(1|32,R)$ transformation and the $u(N)$ transformation. We have a closer look at the mixed symmetry in an attempt to realize the local Lorentz symmetry, and we call such models "gauged models". On the other hand, we call the action (3.15) "nongauged", because we do not mix these two symmetry. This "nongauged symmetry" is close to the symmetry of the IIB matrix model, in which the $SO(9,1)$ Lorentz symmetry and the $U(N)$ gauge symmetry are separate.

In addition, this model incorporates the trivial shift of the supermatrix

$$\delta M = c\mathbf{1}_{N\times N}, \text{ where } c \in osp(1|32,R). \tag{3.17}$$

The action (3.15) is rewritten in terms of the explicit expression of the $osp(1|32,R)$ super Lie algebra $M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}$ as

$$S = -\frac{f_{abc}}{2g^2}(tr(m^a m^b m^c) - 3i\bar{\psi}^a m^b \psi^c) = \frac{i}{g^2} Tr(m_p{}^q [m_q{}^r, m_r{}^p]) - 3i\bar{\psi}[m,\psi]). \tag{3.18}$$

We consider the reduction of the model to the ten-dimensions. To this end, we first rewrite the action (3.15) in terms of the expansion for the eleven-dimensional gamma matrices as (3.9). In order to see the correspondence between the ten-dimensional IIB matrix model, we perform a reduction to the ten dimensions. To this end, we specialize the tenth direction $x^{10} = x^\sharp$, and we retain the $SO(9,1)$ rotational symmetry. We define the ten-dimensional fields by the following chiral decomposition of the bosonic matrix m:

$$\begin{aligned} m &= W\Gamma^\sharp + A_\mu^{(+)}\Gamma^\mu \frac{1+\Gamma^\sharp}{2} + A_\mu^{(-)}\Gamma^\mu \frac{1-\Gamma^\sharp}{2} + \frac{1}{2}C_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{4!}H_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4\sharp} \\ &+ \frac{1}{5!}\left(I^{(+)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}(1+\Gamma^{(+)}) + I^{(-)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}(1+\Gamma^{(-)})\right). \end{aligned} \tag{3.19}$$

Here, we also recall the notation of the left(right)-handed fermion:

$$\psi_L = \frac{1+\Gamma^\sharp}{2}\psi, \quad \psi_R = \frac{1-\Gamma^\sharp}{2}\psi,$$

The coefficients $I^{(+)}_{\mu_1\cdots\mu_5}$ and $I^{(-)}_{\mu_1\cdots\mu_5}$ are self-dual to each other. Namely, they satisfy the relation

$$I^{(\pm)}_{\mu_1\cdots\mu_5} = \frac{-1}{5!} I^{(\mp)}_{\mu_6\cdots\mu_{10}} \epsilon^{\mu_1\cdots\mu_{10}\sharp}. \tag{3.20}$$

The action is thus expanded as

$$\begin{aligned} S_b &= \frac{i}{g^2} Tr_{N\times N}(-96[A^{(+)}_{\mu_1}, A^{(-)}_{\mu_2}]C^{\mu_1\mu_2} - 96W[A^{(+)\mu}, A^{(-)}_\mu] + \frac{4}{5}W[I^{(+)}_{\mu_1\cdots\mu_5}, I^{(-)\mu_1\cdots\mu_5}] \\ &- 4H_{\mu_1\cdots\mu_4}([A^{(+)}_{\mu_5}, I^{(-)\mu_1\cdots\mu_5}] - [A^{(-)}_{\mu_5}, I^{(+)\mu_1\cdots\mu_5}]) - 8C_{\mu_1\mu_2}[I^{(+)\mu_1}{}_{\mu_3\cdots\mu_6}, I^{(-)\mu_2\cdots\mu_6}] \\ &+ \frac{8}{3}H^{\nu\rho}{}_{\mu_1\mu_2}([I^{(+)}{}_{\nu\rho\mu_3\mu_4\mu_5}, I^{(-)\mu_1\cdots\mu_5}] - [I^{(-)}{}_{\nu\rho\mu_3\mu_4\mu_5}, I^{(+)\mu_1\cdots\mu_5}]) \end{aligned}$$

$$+ \quad 32[C^{\mu_1}{}_{\mu_2}, C_{\mu_1\mu_3}]C^{\mu_2\mu_3} - 16C_{\mu_1\mu_2}[H^{\mu_1}{}_{\mu_3\mu_4\mu_5}, H^{\mu_2\cdots\mu_5}] + \frac{1}{27}H_{\mu_1\cdots\mu_4}[H^{\nu}{}_{\mu_5\cdots\mu_7}, H_{\nu\mu_8\mu_9\mu_{10}}]\epsilon^{\mu_1\cdots\mu_{10}\sharp}), \tag{3.21}$$

$$\begin{aligned} S_f &= \frac{i}{g^2}Tr_{N\times N}(-3i(-\bar{\psi}_L[W,\psi_R] + \bar{\psi}_R[W,\psi_L]) - 3i(\bar{\psi}_L\Gamma^{\mu}[A^{(+)}_{\mu},\psi_L] + \bar{\psi}_R\Gamma^{\mu}[A^{(-)}_{\mu},\psi_R]) \\ &- \frac{3i}{2!}(\bar{\psi}_L\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2},\psi_R] + \bar{\psi}_R\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2},\psi_L]) \\ &- \frac{3i}{4!}(-\bar{\psi}_L\Gamma^{\mu_1\mu_2\mu_3\mu_4}[H_{\mu_1\mu_2\mu_3\mu_4},\psi_R] + \bar{\psi}_R\Gamma^{\mu_1\mu_2\mu_3\mu_4}[H_{\mu_1\mu_2\mu_3\mu_4},\psi_L]) \\ &- \frac{3i}{5!}(2\bar{\psi}_L\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}[I^{(+)}{}_{\mu_1\mu_2\mu_3\mu_4\mu_5},\psi_L] + 2\bar{\psi}_R\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}[I^{(-)}{}_{\mu_1\mu_2\mu_3\mu_4\mu_5},\psi_R])), \end{aligned} \tag{3.22}$$

where S_b and S_f are the bosonic and fermionic parts of the action S, respectively. The whole action is $S = S_b + S_f$. The computation of this action is lengthy, and we delegate the proof to Appendix B.2. of [27].

3.2.2 Identification of the supersymmetry with the IIB matrix model

We next discuss the identification of the supersymmetry with that of the IIB matrix model. This model has two kinds of supersymmetry transformation, corresponding to the $osp(1|32,R)$ rotational symmetry and the translation symmetry.

- The former supersymmetry corresponding to the $osp(1|32,R)$ rotation is called the "homogeneous supersymmetry", in the same sense as for the IIB matrix model. This is defined by the rotation by the supercharge $Q = \begin{pmatrix} 0 & \chi \\ i\bar{\chi} & 0 \end{pmatrix}$. Namely, the transformation of the matrices M is given by

$$\delta^{(1)}_{\chi}M = [Q,M] = [\begin{pmatrix} 0 & \chi \\ i\bar{\chi} & 0 \end{pmatrix}, \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}] = \begin{pmatrix} i(\chi\bar{\psi} - \psi\bar{\chi}) & -m\chi \\ i\bar{\chi}m & 0 \end{pmatrix}. \tag{3.23}$$

- The latter supersymmetry corresponds to the translation symmetry. This is called the "inhomogeneous supersymmetry", and is defined by the simple translation of the fermion $\delta^{(2)}_{\xi}\psi = \xi$.

In this way, the $osp(1|32,R)$ supermatrix model has $32+32 = 64$ supercharges. This is twice as much as the IIB matrix model. This simply leads us to speculate that the double structures of the IIB matrix model are embedded in this supermatrix model. We elaborate on this point, by scrutinizing the commutation relation. Especially, we elucidate that there is a nice correspondence with the supersymmetry of the IIB matrix model, with the chiral decomposition (3.19). Namely, we identify $A^{(\pm)}_{\mu}$ with the vector fields of the IIB matrix model, and consider its supersymmetry transformation.

Since the fields $A^{(\pm)}$ are given by[3]

$$A^{(\pm)}_{\mu} = \frac{1}{32}tr(m\Gamma_{\mu}) \mp \frac{1}{32}tr(m\Gamma_{\mu\sharp}), \tag{3.24}$$

the homogeneous transformation of the vector fields $A^{(\pm)}_{\mu}$ and ψ is computed as

$$\delta^{(1)}_{\chi}A^{(+)}_{\mu} = \frac{1}{32}tr((\delta^{(1)}_{\chi}m)\Gamma_{\mu}) + \frac{-1}{32}tr((\delta^{(1)}_{\chi}m)\Gamma_{\mu\sharp}) = \frac{i}{8}\bar{\chi}_R\Gamma_{\mu}\psi_R, \tag{3.25}$$

$$\delta^{(1)}_{\chi}A^{(-)}_{\mu} = \frac{1}{32}tr((\delta^{(1)}_{\chi}m)\Gamma_{\mu}) - \frac{-1}{32}tr((\delta^{(1)}_{\chi}m)\Gamma_{\mu\sharp}) = \frac{i}{8}\bar{\chi}_L\Gamma_{\mu}\psi_L, \tag{3.26}$$

$$\delta^{(1)}_{\chi}\psi = -m\chi. \tag{3.27}$$

The transformation of the vector fields $A^{(\pm)}_{\mu}$ is beautifully reminiscent of the homogeneous transformation of the IIB matrix model. In this sense, we see a correspondence between the chirality of the vector fields and the fermions as

$$A^{(+)}_{\mu} \Leftrightarrow \psi_R, \quad A^{(-)}_{\mu} \Leftrightarrow \psi_L. \tag{3.28}$$

[3] We recall that $\frac{1}{32}tr\Gamma_A\Gamma^A = 1$ while $\frac{1}{32}tr\Gamma_{AB}\Gamma^{AB} = -1$. Here, we do not take a summation with respect to the duplicate indices A, B.

The chirally-decomposed fermions $\psi_{L,R}$ clearly have 16 components, which coincides the number of the supercharges for the IIB matrix model. In this sense, this identification augurs very well for the reproduction of the supersymmetry of the IIB matrix model. On the other hand, the homogeneous transformation of the fermion is given by (3.26), and is devoid of the commutator of the vector fields like the IIB matrix model. This is a downside in the identification of the supersymmetry transformation with that of the IIB matrix model.

Nextly, we have a close look at the commutation relation of the supersymmetry transformation. We wish to find the similar structure of the commutation relations as that of the IIB matrix model. We consider the following commutation relations:

$$(1)[\delta^{(1)}_\chi, \delta^{(1)}_\epsilon]A^{(\pm)}_\mu, \quad (2)[\delta^{(2)}_\chi, \delta^{(2)}_\epsilon]A^{(\pm)}_\mu, \quad (3)[\delta^{(1)}_\chi, \delta^{(2)}_\epsilon]A^{(\pm)}_\mu, \tag{3.29}$$

It is trivial that the commutator (2) vanishes, because the inhomogeneous supersymmetry is nothing but a translation of the fermion. Here, we scrutinize the rest of the commutators one by one.

Supersymmetry transformation $(3)[\delta^{(1)}_\chi, \delta^{(2)}_\epsilon]A^{(\pm)}_\mu$

In the IIB matrix model, only the commutator (2.33) remains nonvanishing. We attempt to extract the similar structure to the commutation relation (2.33). The commutator of the supersymmetry transformation for each m and ψ is given by

$$[\delta^{(1)}_\chi, \delta^{(2)}_\epsilon]m = -i(\chi\bar{\epsilon} - \epsilon\bar{\chi}), \quad [\delta^{(1)}_\chi, \delta^{(2)}_\epsilon]\psi = 0, \tag{3.30}$$

which can be derived in the same way as (2.33). Namely, we take the difference of the following paths:

$$m \overset{\delta^{(2)}_\epsilon}{\to} m \overset{\delta^{(1)}_\chi}{\to} m + i(\chi\bar{\psi} - \psi\bar{\chi}), \text{ whereas } m \overset{\delta^{(1)}_\chi}{\to} m + i(\chi\bar{\psi} - \psi\bar{\chi}) \overset{\delta^{(2)}_\epsilon}{\to} m + i\chi(\bar{\psi} + \bar{\epsilon}) - i(\psi + \epsilon)\bar{\chi},$$
$$\psi \overset{\delta^{(2)}_\epsilon}{\to} \psi + \epsilon \overset{\delta^{(1)}_\chi}{\to} \psi + \epsilon - m\chi, \text{ whereas } \psi \overset{\delta^{(1)}_\chi}{\to} \psi - m\chi \overset{\delta^{(2)}_\epsilon}{\to} \psi + \epsilon - m\chi.$$

The commutator of the supersymmetry transformation for each $A^{(\pm)}_\mu$ can be extracted as

$$[\delta^{(1)}_\chi, \delta^{(2)}_\epsilon]A^{(+)}_\mu = \frac{1}{32}tr(([\delta^{(1)}_\chi, \delta^{(2)}_\epsilon]m)\Gamma_\mu) + \frac{-1}{32}tr(([\delta^{(1)}_\chi, \delta^{(2)}_\epsilon]m)\Gamma_{\mu\sharp}) = \frac{i}{8}\bar{\epsilon}_R\Gamma_\mu\chi_R, \tag{3.31}$$
$$[\delta^{(1)}_\chi, \delta^{(2)}_\epsilon]A^{(-)}_\mu = \frac{1}{32}tr(([\delta^{(1)}_\chi, \delta^{(2)}_\epsilon]m)\Gamma_\mu) - \frac{-1}{32}tr(([\delta^{(1)}_\chi, \delta^{(2)}_\epsilon]m)\Gamma_{\mu\sharp}) = \frac{i}{8}\bar{\epsilon}_L\Gamma_\mu\chi_L. \tag{3.32}$$

Extracting the specific chirality of the SUSY parameters, we obtain the nonvanishing commutators as

$$\begin{aligned} &[\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_L}]A^{(+)}_\mu = 0, && [\delta^{(1)}_{\chi_R}, \delta^{(2)}_{\epsilon_R}]A^{(+)}_\mu = \frac{i}{8}\bar{\epsilon}_R\Gamma_\mu\chi_R, \\ &[\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_L}]A^{(-)}_\mu = \frac{i}{8}\bar{\epsilon}_L\Gamma_\mu\chi_L, && [\delta^{(1)}_{\chi_R}, \delta^{(2)}_{\epsilon_R}]A^{(-)}_\mu = 0. \end{aligned} \tag{3.33}$$

The commutators with different chirality of the two supersymmetry parameters clearly vanish:

$$[\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_R}]A^{(\pm)}_\mu = [\delta^{(1)}_{\chi_R}, \delta^{(2)}_{\epsilon_L}]A^{(\pm)}_\mu = 0. \tag{3.34}$$

These combinations of the supersymmetry transformations clearly resemble the structure of the IIB matrix model. The correspondence between the vector fields and the chirality of the fermionic fields matches that obtained by the correspondence of the homogeneous transformation itself.

Supersymmetry transformation $(1)[\delta^{(1)}_\chi, \delta^{(1)}_\epsilon]A^{(\pm)}_\mu$

In the case of the IIB matrix model, this supersymmetry transformation vanishes on shell up to the gauge transformation, as we have seen in (2.31). We want to find the same vanishing in our cubic model, if we are to identify the supersymmetry transformation. However, it turns out that this commutator does not vanish. In investigating this commutation relation, it is easier to utilize the following identity:

$$[\delta^{(1)}_\chi, \delta^{(1)}_\epsilon]M = [Q_\chi, [Q_\epsilon, M]] - [Q_\epsilon, [Q_\chi, M]] = [[Q_\chi, Q_\epsilon], M]. \tag{3.35}$$

This commutator of the SUSY transformation is thus obtained by

$$[\delta^{(1)}_\chi, \delta^{(1)}_\epsilon]M = [\begin{pmatrix} i(\chi\bar{\epsilon} - \epsilon\bar{\chi}) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} m & \psi \\ i\bar{\psi} & 0 \end{pmatrix}] = \begin{pmatrix} [i(\chi\bar{\epsilon} - \epsilon\bar{\chi}), m] & i(\chi\bar{\epsilon} - \epsilon\bar{\chi})\psi \\ -i\bar{\psi}(\chi\bar{\epsilon} - \epsilon\bar{\chi}) & 0 \end{pmatrix}. \tag{3.36}$$

This relation reveals that the commutators of the two homogeneous transformation are

$$[\delta^{(1)}_{\chi}, \delta^{(1)}_{\epsilon}]m = i[(\chi\bar{\epsilon} - \epsilon\bar{\chi}), m], \tag{3.37}$$

$$[\delta^{(1)}_{\chi}, \delta^{(1)}_{\epsilon}]\psi = i(\chi\bar{\epsilon} - \epsilon\bar{\chi})\psi. \tag{3.38}$$

We have seen in (3.33) which chiralities of the supersymmetry parameters correspond to the vector fields $A^{(\pm)}_{\mu}$. According to this correspondence, we examine the commutation relation with respect to the following cases.

We first investigate the commutation relation $[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_L}]$. We examine this commutation relation of this SUSY transformation:

$$[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_L}]A^{(-)}_{\mu} = \frac{1}{32}tr(([\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_L}]m)\Gamma_{\mu}) - \frac{-1}{32}tr(([\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_L}]m)\Gamma_{\mu\sharp}) = \frac{i}{8}(\bar{\chi}_L[m, \Gamma_{\mu}]\epsilon_L), \tag{3.39}$$

Here, we recall that the commutator $[m, \Gamma_{\mu}]$ belong to the $sp(32)$ Lie algebra (namely, only the rank-1,2,5 contributions remain). Unfortunately, this commutation relation does not vanish exactly in the ten-dimensional reduction. Now, we have a close look at which components survive or vanish: Since the chirality of the supersymmetry parameter is identical, the contribution of the odd-rank fields of m (with respect to the ten-dimensional indices) clearly vanishes. On the other hand, the contribution of the even-rank fields $W, C_{\mu_1\mu_2}, H_{\mu_1\cdots\mu_4}$ survives. For example, the contribution of the rank-0 field W is obtained as

$$\bar{\chi}_L[W\Gamma_{\sharp}, \Gamma_{\mu}]\epsilon_L = \bar{\chi}_L W\Gamma_{\sharp}\Gamma_{\mu}\epsilon_L (\neq 0). \tag{3.40}$$

We next investigate the commutation relation of other chirality. $[\delta^{(1)}_{\chi_R}, \delta^{(1)}_{\epsilon_R}]$. The corresponding field is now $A^{(+)}_{\mu}$, and the commutation relation is now

$$[\delta^{(1)}_{\chi_R}, \delta^{(1)}_{\epsilon_R}]A^{(+)}_{\mu} = \frac{1}{32}tr(([\delta^{(1)}_{\chi_R}, \delta^{(1)}_{\epsilon_R}]m)\Gamma_{\mu}) + \frac{-1}{32}tr(([\delta^{(1)}_{\chi_R}, \delta^{(1)}_{\epsilon_R}]m)\Gamma_{\mu\sharp}) = \frac{i}{8}(\bar{\chi}_R[m, \Gamma_{\mu}]\epsilon_R), \tag{3.41}$$

by completely computing in the same fashion as in (3.39), and we note that the contribution of the even-rank fields $W, C_{\mu_1\mu_2}, H_{\mu_1\cdots\mu_4}$ remains nonvanishing.

Even the worse news comes in the commutator for the different chirality of the supersymmetry parameter. The commutator for the rank-1 vector field is given by

$$[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]A^{(+)}_{\mu} = \frac{1}{32}tr(([\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]m)\Gamma_{\mu}) + \frac{-1}{32}tr(([\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]m)\Gamma_{\mu\sharp}) = \frac{i}{16}(\bar{\chi}_L m\Gamma_{\mu}\epsilon_R + \bar{\epsilon}_R\Gamma_{\mu}m\chi_L), \tag{3.42}$$

$$[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]A^{(-)}_{\mu} = \frac{1}{32}tr(([\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]m)\Gamma_{\mu}) - \frac{-1}{32}tr(([\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]m)\Gamma_{\mu\sharp}) = -\frac{i}{16}(\bar{\epsilon}_R m\Gamma_{\mu}\chi_L + \bar{\chi}_L\Gamma_{\mu}m\epsilon_R). \tag{3.43}$$

The disastrous fact is that these commutators include the fields $A^{(+)}_{\mu}$ and $A^{(-)}_{\mu}$ as

$$[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]A^{(+)}_{\mu} = -\frac{i}{8}\bar{\chi}_L A^{(+)}_{\nu}\Gamma_{\mu}{}^{\nu}\epsilon_R + \cdots, \quad [\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\epsilon_R}]A^{(-)}_{\mu} = -\frac{i}{8}\bar{\chi}_L A^{(-)}_{\nu}\Gamma_{\mu}{}^{\nu}\epsilon_R + \cdots. \tag{3.44}$$

These commutation relations reveal that the two-fold supersymmetry is not independent of each other, but are connected by not the impurity $W, C_{\mu_1\mu_2}$ and $H_{\mu_1\cdots\mu_4}$, but the fields $A^{(\pm)}_{\mu}$. This is an unfavorable situation in the analysis of the supersymmetry transformation of this cubic model.

Summary of the supersymmetry structure

We recapitulate what we have obtained from the above argument of the supersymmetry. Firstly, we have noted that this supermatrix model incorporates the two-fold structure of the supersymmetry of the IIB matrix model. This is predictable from the number of the supercharges. This model has the homogeneous supersymmetry and the inhomogeneous supersymmetry coming from the $osp(1|32, R)$ rotation and the translation symmetry, respectively. Therefore, this model has 64 supercharges, which is twice as many as the IIB matrix model. The IIB-like structure of the supersymmetry is extracted by the chiral decomposition of the rank-1 field and the supercharges, and we have seen the correspondence (3.28). While we have reproduced the commutation relation (2.31) and (2.32) from the commutator $[\delta^{(1)}_{\chi}, \delta^{(2)}_{\epsilon}]$ and $[\delta^{(2)}_{\chi}, \delta^{(2)}_{\epsilon}]$, the commutator $[\delta^{(1)}_{\chi}, \delta^{(1)}_{\epsilon}]$ does not trivially vanish, unlike the IIB matrix model. Moreover, the commutation relation (3.44) unravels that these two supersymmetries are not independent of each other. Its physical interpretation will be reported elsewhere.

3.2.3 Induction of the IIB matrix model

We have seen the correspondence of the supersymmetry with the IIB matrix model, and we have elucidated the correspondence of the chirality between the rank-1 vector fields and the supersymmetry parameter as (3.28). This leads us to the idea that we may be able to induce the IIB matrix model if we integrate out certain fields. To this end, we retain the field $A^{(+)}_\mu$ and ψ_R, and integrate out the other fields. However, the solid discussion for the extraction of the IIB matrix model is an onerous question, and we contend ourselves with a hand-waving argument. This supermatrix model possesses no time spacetime derivative from the outset, as is true of the IIB matrix model. Its classical equation of motion $\partial_A \frac{\partial S}{\partial(\partial_A X)} - \frac{\partial S}{\partial X} = 0$ clearly incorporates the following noncommutative background

$$(A^{(+)}_\mu)_{\text{b.g.}} = \hat{p}_\mu, \quad (\text{other fields}) = 0, \tag{3.45}$$

where $\hat{p}_\mu$ satisfies the canonical commutation relation $[\hat{p}_\mu, \hat{p}_\nu] = ic_{\mu\nu}$. When we denote the fluctuation as $A^{(+)}_\mu = \hat{p}_\mu + a^{(+)}_\mu$, the commutator is mapped as

$$[A^{(+)}_\mu, X] = -i\partial_\mu X + [a^{(+)}_\mu, X].$$

Then, the bosonic and the fermionic terms of the action are expanded as

$$\begin{aligned}
S_b &= \frac{1}{g^2} Tr_{N\times N}(-96(\partial_{\mu_1} A^{(-)}_{\mu_2})C^{\mu_1\mu_2} + 96(\partial_\mu W)A^{(-)\mu} + 4(\partial_{\mu_1} H_{\mu_2\cdots\mu_5})I^{(-)\mu_1\cdots\mu_5}) &(3.46)\\
&+ \frac{i}{g^2} Tr_{N\times N}(-96[a^{(+)}_{\mu_1}, A^{(-)}_{\mu_2}]C^{\mu_1\mu_2} - 96W[a^{(+)\mu}, A^{(-)}_\mu] + \frac{4}{5}W[I^{(+)}_{\mu_1\cdots\mu_5}, I^{(-)\mu_1\cdots\mu_5}]\\
&+ 4([a^{(+)}_{\mu_1}, H_{\mu_2\cdots\mu_5}]I^{(-)\mu_1\cdots\mu_5} - [A^{(-)}_{\mu_1}, H_{\mu_2\cdots\mu_5}]I^{(+)\mu_1\cdots\mu_5}) - 8C_{\mu_1\mu_2}[I^{(+)\mu_1}{}_{\mu_3\cdots\mu_6}, I^{(-)\mu_2\cdots\mu_6}]\\
&+ \frac{8}{3}H^{\nu\rho}{}_{\mu_1\mu_2}([I^{(+)}{}_{\nu\rho\mu_3\mu_4\mu_5}, I^{(-)\mu_1\cdots\mu_5}] - [I^{(-)}{}_{\nu\rho\mu_3\mu_4\mu_5}, I^{(+)\mu_1\cdots\mu_5}])\\
&+ 32[C^{\mu_1}{}_{\mu_2}, C_{\mu_1\mu_3}]C^{\mu_2\mu_3} - 16C_{\mu_1\mu_2}[H^{\mu_1}{}_{\mu_3\mu_4\mu_5}, H^{\mu_2\cdots\mu_5}] + \frac{1}{27}H_{\mu_1\cdots\mu_4}[H^\rho{}_{\mu_5\cdots\mu_7}, H_{\rho\mu_8\mu_9\mu_{10}}]\epsilon^{\mu_1\cdots\mu_{10}\sharp}), &(3.47)\\
S_f &= \frac{1}{g^2} Tr_{N\times N}(-3i\bar{\psi}_L\Gamma^\mu\partial_\mu\psi_L) &(3.48)\\
&+ \frac{i}{g^2} Tr_{N\times N}(-3i(-\bar{\psi}_L[W,\psi_R] + \bar{\psi}_R[W,\psi_L]) - 3i(\bar{\psi}_L\Gamma^\mu[a^{(+)}_\mu, \psi_L] + \bar{\psi}_R\Gamma^\mu[A^{(-)}_\mu, \psi_R])\\
&- \frac{3i}{2!}(\bar{\psi}_L\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2}, \psi_R] + \bar{\psi}_R\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2}, \psi_L])\\
&- \frac{3i}{4!}(-\bar{\psi}_L\Gamma^{\mu_1\mu_2\mu_3\mu_4}[H_{\mu_1\mu_2\mu_3\mu_4}, \psi_R] + \bar{\psi}_R\Gamma^{\mu_1\mu_2\mu_3\mu_4}[H_{\mu_1\mu_2\mu_3\mu_4}, \psi_L])\\
&- \frac{3i}{5!}(2\bar{\psi}_L\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}[I^{(+)}{}_{\mu_1\mu_2\mu_3\mu_4\mu_5}, \psi_L] + 2\bar{\psi}_R\Gamma^{\mu_1\mu_2\mu_3\mu_4\mu_5}[I^{(-)}{}_{\mu_1\mu_2\mu_3\mu_4\mu_5}, \psi_R])). &(3.49)
\end{aligned}$$

The cubic action, per se, does not include a quartic term of vector fields in the action. However, we can interpret that the bosonic term is induced by the fermionic term of the the IIB matrix model. The idea that the theory consisting of fermionic fields and a Dirac operator induces the Einstein gravity and the Yang-Mills theory has long been suggested. The proposal of Connes and Chamseddine is one of these suggestions of the induced gravity [3]. Based on these suggestions, we find it a natural idea that the bosonic term of the IIB matrix model is induced from its fermionic term. And we hypothesize that the whole IIB model should be induced only by the fermionic field.

Our goal is thus to find the fermionic terms in this cubic matrix model to be identified with that of the IIB matrix model. We have seen in the previous section the correspondence between the vector fields and the fermionic fields according to the identification of $\mathcal{N} = 2$ supersymmetry with that of the IIB matrix model. And the fermionic terms to be identified with that if the IIB matrix model is

$$\bar{\psi}_R\Gamma^\mu A^{(+)}_\mu\psi_R \overset{\text{or}}{\Leftrightarrow} \bar{\psi}_L\Gamma^\mu A^{(-)}_\mu\psi_L. \tag{3.50}$$

If either of these terms exists in the action, this can be identified with the fermionic term of the IIB matrix model. However, the kinetic term of the fermionic fields (3.48), per se, does not include such terms as

(3.50), and does not serve to induce the IIB matrix model due to the discrepancy of the correspondence of the vector fields and the chirality of the fermions. In order to remedy this situation, we consider inducing the terms (3.50) by the multi-loop effect. The action (3.46) $\sim$ (3.49) just tells us that there is no terms to be identified with the IIB matrix model *at a tree level*. Since the idea of 'inducing the IIB matrix model' is to construct the bosonic term by the one-loop effect of the fermionic Lagrangian, the idea of 'induced theory' is a notion based on the multi-loop effect of the perturbative theory. In this sense, there is no problem if the fermionic term is induced by the multi-loop effect.

We can read off the Feynman rule from the above action. The typical propagators and the vertices are given below: Our goal is to build a fermion vertex (3.50) by means of the multi-loop effect. In order to

Figure 3: Propagators of this cubic supermatrix model.

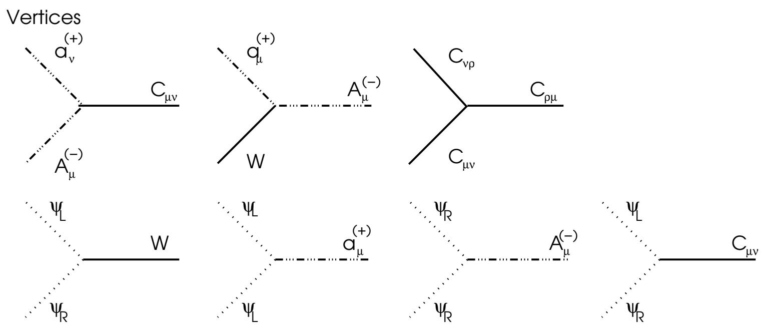

Figure 4: Vertices of this cubic supermatrix model.

build such a term, the above Feynman diagrams do not suffice. Especially we are lacking the propagators of the fields. It is thus necessary to build an induced propagators by means of the multi-loop effect of the existing Feynman rule. These induced propagators are constructed by the following loop effect.

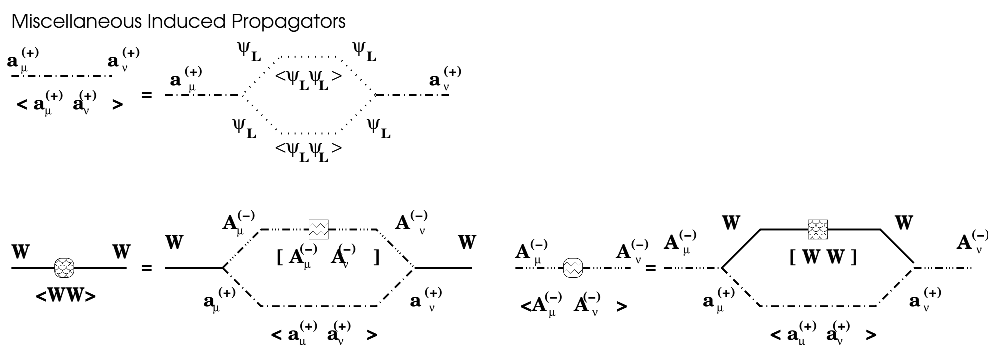

Figure 5: The induced propagators of the bosonic fields

- $\langle a_\mu^{(+)} a_\nu^{(+)} \rangle$: It is easy to construct this induced propagator. All we have to do is to connect the fermions using the existing fermion propagators $\langle \psi_L \psi_L \rangle$.

- $\langle WW \rangle$ and $\langle A_\mu^{(-)} A_\nu^{(-)} \rangle$: The construction of these propagators is a difficult problem. Unfortunately, it is impossible to construct these propagators perturbatively, and we delegate the disproof of their existence in Appendix B.4. of [27]. However, this is not the end of the story. Even if we fail to induce these propagators perturbatively, we have a choice to induce these propagators by means of

the nonperturbative effect. These propagators may be induced by the following recursive structure. This structure is reminiscent of the self-consistency condition of Nambu-Jona-Lasino model. And

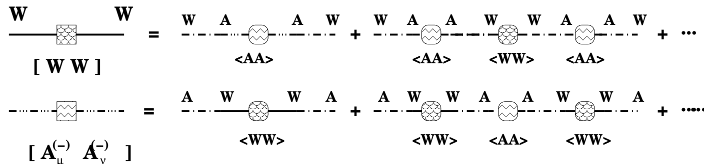

Figure 6: The induced propagators of the bosonic fields

when we consider the nonperturbative[4] effect, there is no particular conservation law which prohibits the existence of the propagators $\langle WW\rangle$ or $\langle A^{(-)}_\mu A^{(-)}_\nu\rangle$. Throughout our discussion, we assume the existence of these propagators.

Now we are ready to construct the induced propagators of this cubic model. In constructing the vertices of the fermionic terms (3.50). The answer is now easy, and the IIB-like vertex is constructed by the following procedure.

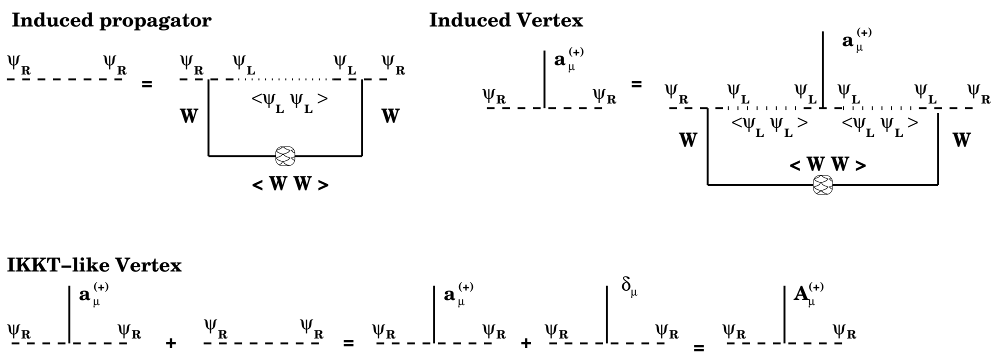

Figure 7: The induced IKKT-like vertex $\bar{\psi}\Gamma^\mu A^{(+)}_\mu\psi$

- To construct the induced vertex corresponding to (3.50), we must first construct two objects. One is the propagator $\langle\psi_R\psi_R\rangle$. This is easily constructed once we admit the existence of the propagator $\langle WW\rangle$.

- Another object is the vertex $\bar{\psi}_R\Gamma^\mu a^{(+)}_\mu\psi_R$. It is also easy to construct this vertex utilizing the induced propagator $\langle WW\rangle$.

- These two objects serve to induce our desired term $\bar{\psi}_R\Gamma^\mu A^{(+)}_\mu\psi_R$. Note that this induced propagator indicates that the kinetic term $\bar{\psi}_R\Gamma^\mu\partial_\mu\psi_R$. This is diagrammatically regarded as the vertex of ψ_R, ψ_R and $\partial_\mu \sim \hat{p}_\mu$, where $\hat{p}_\mu$, around which we have expanded the theory. Therefore, the sum of these two objects is regarded as

$$\langle\psi_R\psi_R\rangle + \bar{\psi}_R\Gamma^\mu a^{(+)}_\mu\psi_R = -i\bar{\psi}_R\partial_\mu\psi_R + \bar{\psi}_R\Gamma^\mu a^{(+)}_\mu\psi_R = \bar{\psi}_R\Gamma^\mu A^{(+)}_\mu\psi_R. \tag{3.51}$$

We thus build the vertex for the bosonic field $A^{(+)}_\mu$ and the corresponding fermionic field ψ_R. This fermionic operator serves to derive the bosonic quartic terms of the IIB matrix model. In this way, we expect that the IIB matrix model is induced from the $osp(1|32,R)$ supermatrix model.

[4]Here, we mean the word 'nonperturbative' by the effect not stemming from the multi-loop effect of Feynman diagram.

3.3 $gl(1|32,R)\otimes gl(N,R)$ gauged cubic matrix model

We next investigate the gauged version of the supermatrix model. Originally, L. Smolin [19] touched on the suggestion to enhance the symmetry of the matrix model, by regarding the tensor product as not for the Lie group but for the Lie algebra. This attempt is fascinating, in that it broadens the symmetry greatly. Smolin's proposal turns out to be essential to build a matrix model with local Lorentz invariance. To elaborate on this point, let us think about the extension of the IIB matrix model to the local Lorentz invariant model. In the IIB matrix model, we regard the eigenvalues of the matrices as the spacetime coordinate. Therefore, in order to render the Lorentz symmetry "local", it is imperative that the parameter of the Lorentz transformation should be dependent on the $u(N)$ element in a nontrivial way [26, 38]. Then, we introduce the so-called "gauged matrix model", in order to build such a matrix model.

In the IIB matrix model, the $SO(9,1)$ Lorentz symmetry and the $SU(N)$ symmetry are totally decoupled. Now, let ξ and u be the generator of the $SO(9,1)$ and the $U(N)$ symmetry, respectively (i.e. $\xi\in so(9,1)$ and $u\in u(N)$). The symmetry is a tensor product of the group; namely they are decoupled in terms of the Lie algebra as

$$\exp(\xi\otimes\mathbf{1}+\mathbf{1}\otimes u)=e^{\xi}\otimes e^{u}. \tag{3.52}$$

This holds of the nongauged version of the supermatrix model.

On the other hand, in the gauged model, we enhance the symmetry drastically by mixing these two symmetry. Namely, the element of the transformation group is not $\exp(\xi\otimes\mathbf{1}+\mathbf{1}\otimes u)$ but

$$\exp(\xi\otimes u). \tag{3.53}$$

In this way, we build a Lorentz symmetry dependent on the $u(N)$ symmetry. We define a generator with the tensor product of the two Lie algebra. However, we need a caution in defining a tensor product of the Lie algebra. Suppose $\mathcal{A},\mathcal{B}$ are two different Lie algebra whose bases are $\{a_i\}$ and $\{b_i\}$, respectively. Generally, the space $\mathcal{A}\otimes\mathcal{B}$, which is spanned by the basis $a_i\otimes b_j$, does not close with respect to the commutator

$$[a_1\otimes b_1,a_2\otimes b_2]=\frac{1}{2}\left([a_1,a_2]\otimes\{b_1,b_2\}+\{a_1,a_2\}\otimes[b_1,b_2]\right),\ \text{for } a_1,a_2\in A,\ \ b_1,b_2\in B. \tag{3.54}$$

Rather, we define the tensor product by $\mathcal{A}\check{\otimes}\mathcal{B}$, which is the smallest Lie algebra that includes $\mathcal{A}\otimes\mathcal{B}$ as a subset[5].

In the large-N reduced models, there are several ways to interpret the spacetime coordinate. In the twisted reduced models [65, 66], the matrices A_μ represent the covariant derivative in the spacetime. On the other hand, in the IIB matrix model, the eigenvalues of the matrices A_μ are regarded as the spacetime coordinate. Both relations are linked by the expansion around the flat noncommutative background

$$A_\mu=\hat{p}_\mu+a_\mu,\ \text{where } [\hat{p}_\mu,\hat{p}_\nu]=ic_{\mu\nu}. \tag{3.55}$$

The IIB matrix model reduces to the non-commutative Yang-Mills theory after this reduction. The fermionic term of the IIB matrix model $\frac{-1}{2g^2}Tr\bar{\psi}\Gamma^\mu[A_\mu,\psi]$ reduced to the action of the fermion in the flat space

$$\int d^dx\bar{\psi}(x)i\Gamma^\mu(\partial_\mu\psi(x)+[a_\mu(x),\psi(x)]). \tag{3.56}$$

[5] To illustrate this idea, we introduce the following simple case. The Lie algebra $su(6)$ is known as the tensor product of the Lie algebras $su(3)$ and $su(2)$:

$$su(6)=su(3)\check{\otimes}su(2).$$

This fact is discerned as follows. Let λ^a and σ^i be the basis of the Lie algebra $su(3)$ and $su(2)$, respectively. $su(3)$ and $su(2)$ are 8 and 3 dimensional Lie algebras respectively, and the indices run $a=1,\cdots,8$ and $i=1,2,3$. The tensor product $su(3)\check{\otimes}su(2)$ consists of the following elements.

- $\lambda^a\otimes\sigma^i$: These are the elements of the tensor product as a set of matrices: $su(3)\otimes su(2)$. This set, per se, does not constitute a closed Lie algebra.
- $\lambda^a\otimes\mathbf{1}$ and $\mathbf{1}\otimes\sigma^i$: These are the generators of the group $SU(3)\times SU(2)$.

These elements are known to constitute the algebra of $su(6)$. This example shows in a pedagogical way how the notion of 'gauged theory' enhances the gauge symmetry. While the Lie algebra of the gauge group $SU(3)\times SU(2)$ is a $8+3=11$ dimensional algebra, the tensor product $su(3)\check{\otimes}su(2)$ is a $8+3+8\times 3=35$ dimensional Lie algebra. This is the structure of the enhancement of the gauge symmetry. Note that the similar enhancement of the gauge symmetry is seen in Smolin's proposal [19, 23].

This reminds us of the correspondence between the differential operators and the matrices.

With this idea in mind, we attempt to formulate a matrix model with the local Lorentz invariance, in which the matrices A_μ look like differential operators. The action should reduce in the classical low-energy limit to the fermionic action on the curved spacetime

$$S_F = \int d^d x e(x)\bar{\psi}(x) i\Gamma^\mu {e_\mu}^i(x)\left(\partial_i \psi(x) + [A_i(x), \psi(x)] + \frac{1}{4}\Gamma^{\nu\rho}\omega_{i\nu\rho}(x)\psi(x)\right). \tag{3.57}$$

Here, the indices $\mu, \nu, \rho, \cdots$ and $i, j, k, \cdots$ both run over $0, 1, \cdots, d-1$. The former and the latter denote the indices of the Minkowskian and the curved spacetime, respectively. In considering the correspondence with the matrix model, we need to absorb $e(x)$ into the definition of the fermionic field, since it is not $\int d^d x e(x)$ but $\int d^d x$ that corresponds to the trace of the large N matrices. And we regard the "spinor root density"

$$\Psi(x) = e^{\frac{1}{2}}(x)\psi(x) \tag{3.58}$$

as the fundamental quantity. Then, the action is rewritten as

$$\begin{aligned}
S_F &= \int d^d x \bar{\Psi}(x) e^{\frac{1}{2}}(x) i\Gamma^\mu {e_\mu}^i(x)\Big\{\partial_i(e^{-\frac{1}{2}}(x)\Psi(x)) + [A_i(x), e^{-\frac{1}{2}}(x)\Psi(x)] \\
&\qquad + \frac{1}{4}\Gamma^{\nu\rho}\omega_{i\nu\rho}(x)e^{-\frac{1}{2}}(x)\Psi(x)\Big\} \\
&= \int d^d x \Big\{\bar{\Psi}(x) i\Gamma^\mu \left[{e_\mu}^i(x)\partial_i + \frac{1}{2}{e_\rho}^i(x)\omega_{i\rho\mu}(x) + {e_\mu}^i(x)e^{\frac{1}{2}}(x)(\partial_i e^{-\frac{1}{2}}(x))\right]\Psi(x) \\
&\qquad + i\bar{\Psi}(x)\Gamma^\mu {e_\mu}^i(x)[A_i(x), \Psi(x)] + \frac{i}{4}\bar{\Psi}(x)\Gamma^{\mu_1\mu_2\mu_3}{e_{[\mu_1}}^i(x)\omega_{i\mu_2\mu_3]}(x)\Psi(x)\Big\} \\
&= \int d^d x \Big\{\bar{\Psi}(x) i\Gamma^\mu {e_\mu}^i(x)(\partial_i\Psi(x) + [A_i(x), \bar{\Psi}(x)]) + \frac{i}{4}\bar{\Psi}(x)\Gamma^{\mu_1\mu_2\mu_3}{e_{[\mu_1}}^i(x)\omega_{i\mu_2\mu_3]}(x)\Psi(x)\Big\},
\end{aligned} \tag{3.59}$$

where we have utilized in the last equality the fact that the fermionic field $\Psi(x)$ is Majorana, which leads to the cancellation $\bar{\Psi}(x)\Gamma^\mu\Psi(x) = 0$. The corresponding matrix model is formulated as

$$S'_F = -\frac{1}{2}Tr\bar{\psi}\Gamma^\mu[A_\mu, \psi] - \frac{i}{2}Tr\bar{\psi}\Gamma^{\mu_1\mu_2\mu_3}\{A_{\mu_1\mu_2\mu_3}, \psi\}. \tag{3.60}$$

In promoting the action (3.59) to the matrix model, we have identified the covariant derivative with the commutator with the rank-1 matrix A_μ. The rank-3 term is a naive product of the spin connection and the fermion, and it is natural to promote the product to the anticommutator of the large N hermitian matrices[6]. In the rank-3 term, the pure imaginary number i is necessary so that the action should be hermitian. The action (3.60) is an addition of the rank-3 term to the fermionic term of the action of IIB matrix model. Especially when ψ is a Majorana fermion, (3.60) is equivalent to the following action:

$$S''_F = -Tr\bar{\psi}(\Gamma^\mu A_\mu + i\Gamma^{\mu_1\mu_2\mu_3}A_{\mu_1\mu_2\mu_3})\psi. \tag{3.61}$$

The equivalence between (3.60) and (3.61) is verified as

$$\begin{aligned}
S'_F &= -\frac{1}{2}Tr\bar{\psi}\Gamma^\mu[A_\mu, \psi] - \frac{i}{2}Tr\bar{\psi}\Gamma^{\mu_1\mu_2\mu_3}\{A_{\mu_1\mu_2\mu_3}, \psi\} \\
&= -\frac{1}{2}\bar{\psi}^A\Gamma^\mu A^B_\mu\psi^C Tr(t^A[t^B, t^C]) - \frac{i}{2}\bar{\psi}^A\Gamma^{\mu_1\mu_2\mu_3}A^B_{\mu_1\mu_2\mu_3}\psi^C Tr(t^A\{t^B, t^C\})
\end{aligned}$$

[6]For the readers' convenience, we summarize the commutation relations of the hermitian and anti-hermitian operators. Let $\mathbf{H}$ and $\mathbf{A}$ be the set of the hermitian and anti-hermitian operators, respectively. When $h, h_1, h_2 \in \mathbf{H}$ and $a, a_1, a_2 \in \mathbf{A}$, their commutation relations are as follows:

$$[h_1, h_2] \in \mathbf{A}, \quad [h, a] \in \mathbf{H}, \quad [a_1, a_2] \in \mathbf{A}, \quad \{h_1, h_2\} \in \mathbf{H}, \quad \{h, a\} \in \mathbf{A}, \quad \{a_1, a_2\} \in \mathbf{H}.$$

We prove only the first relation, because the others are easily derived in the same way:

$$[h_1, h_2]^\dagger = (h_1h_2 - h_2h_1)^\dagger = h_2^\dagger h_1^\dagger - h_1^\dagger h_2^\dagger = h_2h_1 - h_1h_2 = -[h_1, h_2].$$

$$
\begin{aligned}
&= -Tr(t^A t^B t^C)\left(\frac{1}{2}(\bar{\psi}^A\Gamma^\mu A^B_\mu\psi^C - \bar{\psi}^C\Gamma^\mu A^B_\mu\psi^A)\right. \\
&\qquad\qquad \left. +\frac{i}{2}(\bar{\psi}^A\Gamma^{\mu_1\mu_2\mu_3}A^B_{\mu_1\mu_2\mu_3}\psi^C + \bar{\psi}^C\Gamma^{\mu_1\mu_2\mu_3}A^B_{\mu_1\mu_2\mu_3}\psi^A)\right) \\
&= -Tr\bar{\psi}(\Gamma^\mu A_\mu + i\Gamma^{\mu_1\mu_2\mu_3}A_{\mu_1\mu_2\mu_3})\psi.
\end{aligned}
$$

Let us next consider the local Lorentz transformation of this matrix model. Originally the local Lorentz transformation of the fermionic field $\delta\psi(x) = \frac{1}{4}\Gamma^{\mu_1\mu_2}\varepsilon_{\mu_1\mu_2}(x)\psi(x)$ should be promoted to the anticommutator of the hermitian matrices as

$$
\delta\psi \stackrel{?}{=} \frac{1}{4}\Gamma^{\mu_1\mu_2}\{\varepsilon_{\mu_1\mu_2}, \psi\}, \tag{3.62}
$$

in order to retain the hermiticity. However, it is a very onerous problem to find an action invariant under this transformation. The conundrum stems from the *noncommutativity of the matrices.* The local Lorentz transformation of the fermion (3.62) with respect to the action (3.61) entails the terms

$$
\delta S''_F = -\frac{1}{4}Tr\bar{\psi}\Gamma^\mu A_\mu \Gamma^{\nu_1\nu_2}\psi\varepsilon_{\nu_1\nu_2} + \cdots. \tag{3.63}
$$

It is extremely difficult to absorb this transformation via the local Lorentz transformation of the bosonic fields A_μ.

This obstacle forces us to abstain the hermiticity of the matrices. We instead promote this transformation to the matrix version as

$$
\delta\psi = \frac{1}{4}\Gamma^{\mu_1\mu_2}\varepsilon_{\mu_1\mu_2}\psi, \tag{3.64}
$$

and we take the action (3.61). Since the fermion is no longer hermitian, the actions (3.60) and (3.61) are no longer equivalent. There are two prices we must pay for this alteration. One is that the product $A_\mu\psi$ does not directly correspond to the covariant derivative. The other is that the matrix model (3.61) is no longer invariant under the translation $\delta A_\mu = c_\mu \mathbf{1}$, which is a spacetime translation in the original interpretation of the IIB matrix model.

The local Lorentz transformation of the action (3.61) is

$$
\delta S''_F = \frac{1}{4}Tr\bar{\psi}\,[\Gamma^\mu A_\mu + i\Gamma^{\mu_1\mu_2\mu_3}A_{\mu_1\mu_2\mu_3}, \Gamma^{\nu_1\nu_2}\varepsilon_{\nu_1\nu_2}]\,\psi. \tag{3.65}
$$

Since we are now considering the local Lorentz transformation and their space-time dependent parameters are promoted to $u(N)$ matrices, the infinitesimal parameters $\varepsilon_{\mu\nu}$ are $u(N)$ matrices. Then, the commutator

$$
[i\Gamma^{\mu_1\mu_2\mu_3}A_{\mu_1\mu_2\mu_3}, \Gamma^{\nu_1\nu_2}\varepsilon_{\nu_1\mu_2}] = \frac{i}{2}\underbrace{[\Gamma^{\mu_1\mu_2\mu_3}, \Gamma^{\nu_1\nu_2}]}_{\text{rank-3}}\{A_{\mu_1\mu_2\mu_3}, \varepsilon_{\nu_1\nu_2}\} + \frac{i}{2}\underbrace{\{\Gamma^{\mu_1\mu_2\mu_3}, \Gamma^{\nu_1\nu_2}\}}_{\text{rank-1, 5}}[A_{\mu_1\mu_2\mu_3}, \varepsilon_{\nu_1\nu_2}] \tag{3.66}
$$

must include the rank-5 gamma matrices, and the action (3.61) is not invariant under the local Lorentz transformation.

The similar thing holds true of the generator of the local Lorentz transformation (3.64). In order for the algebra to close, only the rank-2 terms are not sufficient, because the commutator

$$
[\Gamma^{\mu_1\mu_2}\varepsilon_{\mu_1\mu_2}, \Gamma^{\nu_1\nu_2}\varepsilon'_{\nu_1\nu_2}] = \frac{1}{2}\underbrace{[\Gamma^{\mu_1\mu_2}, \Gamma^{\nu_1\nu_2}]}_{\text{rank-2}}\{\varepsilon_{\mu_1\mu_2}, \varepsilon'_{\nu_1\nu_2}\} + \frac{1}{2}\underbrace{\{\Gamma^{\mu_1\mu_2}, \Gamma^{\nu_1\nu_2}\}}_{\text{rank-0, 4}}[\varepsilon_{\mu_1\mu_2}, \varepsilon'_{\nu_1\nu_2}] \tag{3.67}
$$

includes the rank-4 gamma matrices.

Let us recapitulate what we have obtained through this observation.

- Since the information of the spacetime is encoded in the bosonic matrices A_μ, the parameter of the local Lorentz transformation must depend on the $u(N)$ element in a nontrivial manner. This leads us to consider the "gauged" action.

- The correspondence of the matrices A_μ and the covariant derivative motivates us to construct a matrix model in which the matrices A_μ looks like a differential operators.

- It is difficult to retain the hermiticity of the fermionic field, when we build a local Lorentz invariant action. This forces us to abstain the hermiticity of the fields, despite the following drawbacks.
  - ⋆ The term $\bar{\psi}\Gamma^\mu A_\mu \psi$ no longer corresponds to the covariant derivative $\bar{\psi}(x)(\partial_\mu \psi(x) + [a_\mu(x), \psi(x)])$.
  - ⋆ The matrix model (3.61) is no longer invariant under the translation $\delta A_\mu = c_\mu \mathbf{1}$, which is a spacetime translation in the original interpretation of the IIB matrix model

- The "gauged" symmetry necessitates the higher-rank fields and the local Lorentz transformation parameters.

This observation leads us to regard the "gauged" version of the supermatrix model as a mesmerizing candidate for the local Lorentz invariant extension of the IIB matrix model. To this end, we consider the $gl(1|32,R)$ super Lie algebra, which is the analytic continuation of the $u(1|16,16)$, the complexification of the $osp(1|32,R)$. In addition, we mix the $gl(1|32,R)$ symmetry and the $gl(N)$ gauge symmetry.

This observation is also elaborated in another angle without the supermatrix models in Section 4.

3.3.1 Definition of the $u(1|16,16)$ and $gl(1|32,R)$ super Lie algebra

The $gl(1|32,R)$ super Lie algebra is a cousin of the $u(1|16,16)$ super Lie algebra. More accurately, $gl(1|32,R)$ is obtained by the analytic continuation of the imaginary part of the $u(1|16,16)$. Therefore, we start with introducing the $u(1|16,16)$ super Lie algebra.

The very definition of this super Lie algebra is that

$$\text{If } M \in u(1|16,16), \text{ then } M^\dagger G + GM = 0 \text{ for } G = \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix}. \tag{3.68}$$

That this is a complexification of $osp(1|32,R)$ can be seen from the following aspect. Unlike the $osp(1|32,R)$ super Lie algebra, we do not restrict M to be a real supermatrix, where the reality of the supermatrix is defined as $M = M^* = ({}^T M)^\dagger$. Therefore, we must replace the *transpose* by the *hermitian conjugate*. Note that in the real version the complex conjugate is equivalent to the transpose according to the property in Appendix. A.2.

We can confirm that the super Lie algebra $u(1|16,16)$ actually closes in totally the same fashion as in $osp(1|32,R)$. The legitimacy of the metric G also stems from the same logic as in $osp(1|32,R)$. We specify the element of $u(1|16,16)$ group according to the above definition. The result is that

$$\text{If } M \in u(1|16,16), \text{ then } M = \begin{pmatrix} m & \psi \\ i\bar{\psi} & v \end{pmatrix}. \tag{3.69}$$

- v is restricted to be a pure imaginary number.

- u_{A_1} ,$u_{A_1A_2}$ and $u_{A_1\cdots A_5}$ are real numbers, while u, $u_{A_1A_2A_3}$ and $u_{A_1\cdots A_4}$ are pure imaginary numbers.

This result is derived from the very definition of the super Lie algebra $u(1|16,16)$. The complex conjugate of supermatrices is defined in Appendix. A.2:

$$\begin{aligned} M^\dagger G + GM &= \begin{pmatrix} m^\dagger & \phi \\ \psi^\dagger & v^\dagger \end{pmatrix} \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix} + \begin{pmatrix} \Gamma^0 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} m & \psi \\ \phi^\dagger & v \end{pmatrix} \\ &= \begin{pmatrix} m^\dagger\Gamma^0 + \Gamma^0 m & i\phi + \Gamma^0\psi \\ \psi^\dagger\Gamma^0 + i\phi^\dagger & i(v+v^\dagger) \end{pmatrix} = 0. \end{aligned} \tag{3.70}$$

- Since $v + v^\dagger = 0$, (3.70) immediately constrains v to be a pure imaginary number.

- We first investigate the constraint of the bosonic matrix m. These are decomposed as (3.12), like the $osp(1|32,R)$ super Lie algebra. We use the relationship of the gamma matrices $\Gamma^{0T}\Gamma^{\mu_1\cdots\mu_k}\Gamma^0 = (-1)^{\frac{(k+2)(k-1)}{2}}\Gamma^{\mu_1\cdots\mu_k}$. Therefore, $(u_{A_1\cdots A_k})^* = (-1)^{\frac{(k+2)(k-1)}{2}} u_{A_1\cdots A_k}$. This is real for $k = 1, 2, 5$ and pure imaginary for $k = 0, 3, 4$.

- We investigate the relationship between two fermions ψ and $\phi^\dagger$ utilizing the result $i\phi + \Gamma^0\psi = 0$:

$$\phi^\dagger = (i\Gamma^0\psi)^\dagger = (-i)\psi^\dagger({}^T\Gamma^0) = (-i)\psi^\dagger(-\Gamma^0) = i\bar{\psi}. \tag{3.71}$$

We can verify that this is consistent with the condition $\psi^\dagger\Gamma^0 + i\phi^\dagger = 0$.

We are thus finished with the determination of the elements of $u(1|16,16)$ super Lie algebra.

The important property of $u(1|16,16)$ super Lie algebra is that these can be uniquely decomposed into the direct sum of two different representations of $osp(1|32,R)$. We introduce two different representations of $osp(1|32,R)$ super Lie algebra

$$\clubsuit \quad \mathcal{H} = \{M = \begin{pmatrix} m_h & \psi_h \\ i\bar{\psi}_h & 0 \end{pmatrix} | m_h = u_{A_1}\Gamma^{A_1} + \frac{1}{2!}u_{A_1A_2}\Gamma^{A_1A_2} + \frac{1}{5!}u_{A_1\cdots A_5}\Gamma^{A_1\cdots A_5}, \quad u_{A_1}, u_{A_1A_2}, u_{A_1\cdots A_5}, \psi_h \in \mathcal{R}\}, \tag{3.72}$$

$$\clubsuit \quad \mathcal{A}' = \{M = \begin{pmatrix} m_a & i\psi_a \\ \bar{\psi}_a & iv \end{pmatrix} | m_a = u + \frac{1}{3!}u_{A_1A_2A_3}\Gamma^{A_1A_2A_3} + \frac{1}{4!}u_{A_1\cdots A_4}\Gamma^{A_1\cdots A_4}, \quad u, u_{A_1A_2A_3}, u_{A_1\cdots A_4}, i\psi_a, iv \in (\text{pure imaginary})\}. \tag{3.73}$$

And let H and A' be the element of $\mathcal{H}$ and $\mathcal{A}'$ respectively. Because these are real (pure imaginary), these elements respectively satisfy $H^\dagger = {}^TH'$ and $A'^\dagger = -{}^TA'$. Therefore, these elements satisfy the following property

$${}^THG + GH = 0, \text{ for } H \in \mathcal{H}, \quad {}^TA'G - GA' = 0, \text{ for } A' \in \mathcal{A}'. \tag{3.74}$$

The set $\mathcal{H}$ is, by definition, the $osp(1|32,R)$ super Lie algebra itself. We investigate an important property of the subset of these two subalgebras. The commutation and anti-commutation relations are properties of grave importance in getting acquainted with their algebras.

$$\begin{aligned} &(1)[H_1,H_2] \in \mathcal{H}, \quad (2)[H,A'] \in \mathcal{A}', \quad (3)[A'_1,A'_2] \in \mathcal{H}, \\ &(4)\{H_1,H_2\} \in \mathcal{A}', \quad (5)\{H,A'\} \in \mathcal{H}, \quad (6)\{A'_1,A'_2\} \in \mathcal{A}'. \end{aligned} \tag{3.75}$$

where $H, H_1, H_2 \in \mathcal{H}$ and $A', A'_1, A'_2 \in \mathcal{A}'$.
Here, we only give the proof of the first property, because the others can be easily verified likewise.

$$\begin{aligned} {}^T[H_1,H_2]G &= {}^TH_2{}^TH_1G - {}^TH_1{}^TH_2G = {}^TH_2(-GH_1) - {}^TH_1(-GH_2) = GH_2H_1 - GH_1H_2 \\ &= -G[H_1,H_2]. \end{aligned} \tag{3.76}$$

Utilizing these relations, we can discern that $\mathcal{A}'$, as well as $\mathcal{H}'$ is the representations of $osp(1|32,R)$ and also that the algebra $\mathcal{A}'$ is a representation of $osp(1|32,R)$ super Lie algebra by the commutation relation $(2)[H,A'] \in \mathcal{A}'$ for $H \in \mathcal{H}$ and $A' \in \mathcal{A}'$. This commutation relation states that A' remain in the super Lie algebra $\mathcal{A}'$ after the infinitesimal translation by the elements $H \in \mathcal{H}$. In this sense, we can understand that $\mathcal{A}'$ is another representation of $osp(1|32,R)$.

The introduction of these two representations of $osp(1|32,R)$ teaches us the relationship of $osp(1|32,R)$ and $u(1|16,16)$ super Lie algebras. $\mathcal{H}(= osp(1|32,R))$ is a real part of $u(1|16,16)$ Lie algebra, while $\mathcal{A}'$ is its imaginary part. It is clear that the elements of $u(1|16,16)$ can be uniquely decomposed into the direct sum of $\mathcal{H}$ and $\mathcal{A}'$.

$$u(1|16,16) \equiv \mathcal{H} \oplus \mathcal{A}', \tag{3.77}$$

where $\oplus$ denotes the direct sum of two sets.

Nextly, we introduce the $gl(1|32,R)$ super Lie algebra, especially paying attention to the relation with the $u(1|16,16)$. The definition of $gl(1|32,R)$ super Lie algebra is, per se, simple:

$$\clubsuit \quad \text{If } M \in gl(1|32,R), \quad \text{then } M = \begin{pmatrix} m & \psi \\ i\bar{\phi} & v \end{pmatrix}. \tag{3.78}$$

- m is an element of the Lie algebra $gl(32,R)$, id est, m is allowed to be *an arbitrary* 32×32 *bosonic matrix.* Decomposing this by the gamma matrices, this can be expressed by $m = \sum_{k=0}^{5}\frac{1}{k!}u_{A_1\cdots A_k}\Gamma^{A_1\cdots A_k}$, where the coefficients $u_{A_1\cdots A_k}$ are all real.

- ψ and ϕ are independent fermionic vectors. Each of them possesses 32 components, and the components are fermionic real number.
- v is also a real number.

The definition of $gl(1|32,R)$ states nothing. This definition just states that an arbitrary real 33×33 supermatrix is an eligible member of the super Lie algebra $gl(1|32,R)$. Although this definition does not give any restriction to the elements, the correspondence with the complex group $u(1|16,16)$ is an interesting aspect of $gl(1|32,R)$ super Lie algebra. Since ψ and ϕ are independent fermionic vectors, these can be rewritten as

$$\psi=\psi_1+\psi_2,\quad \phi=\psi_1-\psi_2. \tag{3.79}$$

And the bosonic 32×32 matrices are separated by $m=m_1+m_2$, where m_1 is rank-1,2,5 and m_2 is rank-0,3,4. Then, we define the sets $\mathcal{A}$ as follows.

$$\clubsuit\quad \mathcal{A}=\{M=\left(\begin{array}{cc} m_2 & \psi_2 \\ -i\bar{\psi}_2 & v \end{array}\right)|m_2=u+\frac{1}{3!}u_{A_1A_2A_3}\Gamma^{A_1A_2A_3}+\frac{1}{4!}u_{A_1\cdots A_4}\Gamma^{A_1\cdots A_4},$$
$$u,u_{A_1A_2A_3},u_{A_1\cdots A_4},\psi_2,v\in\mathcal{R}\}.$$

The super Lie algebra $gl(1|32,R)$ is clearly the direct sum of these two super Lie algebras

$$gl(1|32,R)=\mathcal{H}\oplus\mathcal{A}. \tag{3.80}$$

These two subalgebras are also the two different representations of $osp(1|32,R)$ super Lie algebra. $\mathcal{H}$ is $osp(1|32,R)$ itself, and the same super Lie algebra as in introduced in $u(1|16,16)$. On the other hand, the subalgebra $\mathcal{A}$ is $\mathcal{A}'=i\mathcal{A}$.[7] And the elements of these subalgebras readily satisfy

$${}^THG+GH=0,\ \text{for } H\in\mathcal{H},\quad {}^TAG-GA=0,\ \text{for } A\in\mathcal{A}. \tag{3.81}$$

And it is clear that these two subalgebras obey totally the same commutation relations as those of $\mathcal{H}$ and $\mathcal{A}'$:

$$\begin{array}{lll} (1)[H_1,H_2]\in\mathcal{H}, & (2)[H,A]\in\mathcal{A}, & (3)[A_1,A_2]\in\mathcal{H}, \\ (4)\{H_1,H_2\}\in\mathcal{A}, & (5)\{H,A\}\in\mathcal{H}, & (6)\{A_1,A_2\}\in\mathcal{A}, \end{array} \tag{3.82}$$

where $H,H_1,H_2\in\mathcal{H}$ and $A,A_1,A_2\in\mathcal{A}$. The proof is completely the same as that of (3.75), and we do not repeat it. The commutation relation $[\mathcal{H},\mathcal{A}]\in\mathcal{A}$ indicates that $\mathcal{A}$ is a representation of $osp(1|32,R)$ super Lie algebra.

Now, the relationship of the three super Lie algebras $osp(1|32,R)$, $u(1|16,16)$ and $gl(1|32,R)$ is clear. We have seen that both $u(1|16,16)$ and $gl(1|32,R)$ are represented by the direct sum of two different representations of $osp(1|32,R)$:

$$u(1|16,16)=\mathcal{H}\oplus\mathcal{A}',\quad gl(1|32,R)=\mathcal{H}\oplus\mathcal{A}. \tag{3.83}$$

The relationship between $\mathcal{A}$ and $\mathcal{A}'$ is

$$\mathcal{A}'=i\mathcal{A}\Rightarrow\ \text{If } A\in\mathcal{A},\ \text{then } iA\in\mathcal{A}'. \tag{3.84}$$

In this sense, we can regard $gl(1|32,R)$ super Lie algebra as *the analytic continuation* of $u(1|16,16)$. Although we adopt a matrix theory with the gauge symmetry $gl(1|32,R)$ unlike Smolin's original proposal [23], we note that $gl(1|32,R)$ is a cousin of the original $u(1|16,16)$.

Nextly, we explain why we have to introduce the complexification of the $osp(1|32,R)$ from the outset. It turns out that the simple tensor product $osp(1|32,R)\otimes u(N)$ does not close with respect to the commutator in mixing these two symmetries. Using the relation (3.75), we obtain the following commutation relation:

$$[(\mathcal{H}\otimes\mathbf{H}),(\mathcal{H}\otimes\mathbf{H})]=(\{\mathcal{H},\mathcal{H}\}\otimes[\mathbf{H},\mathbf{H}])+([\mathcal{H},\mathcal{H}]\otimes\{\mathbf{H},\mathbf{H}\})=(\mathcal{A}'\otimes\mathbf{A})+(\mathcal{H}\otimes\mathbf{H}). \tag{3.85}$$

[7] This means that, if $A\in\mathcal{A}$, then $iA\in\mathcal{A}'$.

Therefore, the tensor product $osp(1|32,R)\otimes u(N) = \mathcal{H}\otimes \mathbf{H}$ is not a closed Lie algebra. In order to build a closed Lie algebra, we are urged to include $\mathcal{A}'\otimes \mathbf{A}$. This satisfies the commutation relation

$$[(\mathcal{A}'\otimes \mathbf{A}),(\mathcal{A}'\otimes \mathbf{A})] = (\{\mathcal{A}',\mathcal{A}'\}\otimes [\mathbf{A},\mathbf{A}]) + ([\mathcal{A}',\mathcal{A}']\otimes \{\mathbf{A},\mathbf{A}\}) = (\mathcal{A}'\otimes \mathbf{A}) + (\mathcal{H}\otimes \mathbf{H}). \tag{3.86}$$

Therefore, we establish the smallest closed Lie algebra that include $osp(1|32,R)\otimes u(N)$ as a subset as

$$osp(1|32,R)\check{\otimes} u(N) = (\mathcal{H}\otimes \mathbf{H}) + (\mathcal{A}'\otimes \mathbf{A}). \tag{3.87}$$

This is analogous to the situation in which we had to introduce the higher-odd-rank fields in the action and the higher-even-rank fields as a parameter of the local Lorentz transformation once we mix the symmetries, as we have seen in (3.66) and (3.67).

Here, instead of promoting each element to a hermitian matrix, we can make a closed Lie algebra by restricting them to real matrices. It is clear that $(\mathcal{H}+\mathcal{A})\otimes gl(N,R) = gl(1|32,R)\otimes gl(N,R)$ forms another closed algebra. In this case, we embed the spacetime into real matrices, instead of hermitian matrices.

3.3.2 Action of $gl(1|32,R)\otimes gl(N,R)$ supermatrix model

The next job is to investigate the action of this theory. The basic idea is similar to $osp(1|32,R)$ non-gauged cubic matrix model, and we proceed rather quickly. The action is

$$\begin{aligned} S &= \frac{1}{g^2} Tr_{N\times N} \sum_{Q,R=1}^{33} ((\sum_{p=1}^{32} {M_p}^Q {M_Q}^R {M_R}^p) - {M_{33}}^Q {M_Q}^R {M_R}^{33}) = \frac{1}{g^2} Tr_{N\times N}(Str_{33\times 33} M^3) \\ &= \frac{1}{g^2} \sum_{a,b,c=1}^{N^2} Str(M^a M^b M^c) Tr(T^a T^b T^c). \end{aligned} \tag{3.88}$$

M is now a multiplet of $gl(1|32,R)$ super Lie algebra, with each component promoted to the element of $gl(N,R)$ Lie algebra. The indices $P,Q,R,\cdots$ runs $P,Q,R,\cdots = 1,\cdots,33$, while $p,q,r,\cdots = 1,\cdots,32$.

We have promoted the 33×33 matrix M to a large $33N\times 33N$ matrices. However, the structure of the promotion is completely different from $osp(1|32,R)$ model. The gauge transformation is with respect to not the separate $gl(1|32,R)$ and $gl(N,R)$, but the tensor product of the Lie algebra $gl(1|32,R)\otimes gl(N,R)$. This drastically enhances the gauge symmetry of the theory. The gauge transformation is thus $M \Rightarrow M + [u,M]$ for an arbitrary element of $u \in gl(1|32,R)\otimes gl(N,R)$.

We can express this supermatrix model in terms of the basis of the $gl(N,R)$. Here, M_P^Q is expanded as $M_P^Q = \sum_{a=1}^{N} (M^a)_P^Q T^a$. Then, the trace is written as

$$Tr(T^a T^b T^c) = \frac{1}{2} Tr(T^a [T^b,T^c]) + \frac{1}{2} Tr(T^a\{T^b,T^c\}) = \frac{1}{4}(f_{abc}+d_{abc}). \tag{3.89}$$

Here, the structure constant f_{abc} and d_{abc} are real. Using this decomposition, we can rewrite the action (3.88) as

$$S = \frac{1}{4g^2}(f_{abc}+d_{abc}) Str(M^a M^b M^c). \tag{3.90}$$

This supermatrix model can be expressed using its explicit expression (3.78) as

$$S = \frac{1}{g^2} Tr(tr(m^3) - 3i\bar{\phi} m\psi - 3i\bar{\phi}\psi v - v^3). \tag{3.91}$$

3.3.3 Wigner-Inönü contraction and supersymmetry

In considering the relation to the IIB matrix model, we need to reduce the model to the ten dimensions. However, this matrix model has a grave difference from the $osp(1|32,R)$ nongauged supermatrix. Since the action (3.88) is no longer based on the commutator for the $gl(N|R)$ matrices, it is not invariant under the inhomogeneous supersymmetry, namely the translation of the fermion. Nevertheless, the matrix model (3.88) accommodates $32+32 = 64$ supercharges. This leads us to speculate that this model may also have the two-fold supersymmetry structure of the IIB-like supersymmetry.

Here, we reduce the model to the ten dimensions by the Wigner-Inönü contraction, in order to extract the translation symmetry. Just as we perceive the earth as a flat two-dimensional space because the earth is much bigger than we are, we consider the physics in apparently 'ten-dimensional' space because of the large radius of the hyperboloid. To this end, we add the linear term of M to the action (3.88) and start from

$$S = \frac{1}{3} Tr_{N\times N} Str(M_t^3) - R^2 Tr_{N\times N} Str M_t. \tag{3.92}$$

By adding the linear term, the matrix model (3.92) incorporates the classical solution[8].

$$\langle M \rangle = \left(\begin{array}{cc} R\Gamma^{\sharp} \otimes \mathbf{1}_{N\times N} & 0 \\ 0 & R \otimes \mathbf{1}_{N\times N} \end{array} \right), \tag{3.93}$$

The expansion around this classical solution amounts to the Wigner-Inönü contraction. We separate the original matrix between the classical solution and the fluctuation and the classical solution as

$$M_t = \langle M \rangle + M = \left(\begin{array}{cc} R\Gamma^{\sharp} & 0 \\ 0 & R \end{array} \right) + \left(\begin{array}{cc} m & \psi \\ i\bar{\phi} & v \end{array} \right). \tag{3.94}$$

Then, the action is

$$I = \frac{1}{3} tr((m + R\Gamma^{\sharp})^3) - i(\bar{\phi} m \psi + \bar{\phi}\psi v + R\bar{\phi}(1+\Gamma^{\sharp})\psi) - \frac{(v+R)^3}{3} - R^2(tr(m + R\Gamma^{\sharp}) - v). \tag{3.95}$$

We ignore the terms of $\mathcal{O}(R^3)$, because this is just a constant. The terms of $\mathcal{O}(R^2)$ vanish, because this is a linear term with respect to the fluctuation. Then, the action is expressed as follows:

$$S = R(tr(m^2\Gamma^{\sharp}) - v^2 - i\bar{\phi}(1+\Gamma^{\sharp})\psi) + \frac{1}{3} tr(m^3) - \frac{v^3}{3} - i(\bar{\phi} m\psi + v\bar{\phi}\psi). \tag{3.96}$$

We divide the bosonic fluctuation m into $m = m_e + m_o$, where m_e and m_o consist of the even-rank and the odd-rank components in terms of the ten dimensions. Namely, they should be explicitly expressed as

$$\begin{aligned} m_e &= Z\mathbf{1} + W\Gamma^{\sharp} + \frac{1}{2}(C_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2} + D_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2\sharp}) + \frac{1}{4!}(G_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4} + H_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4\sharp}), && (3.97) \\ m_o &= \frac{1}{2}(A^{(+)}_{\mu}\Gamma^{\mu}(1+\Gamma^{\sharp}) + A^{(-)}_{\mu}\Gamma^{\mu}(1-\Gamma^{\sharp})) \\ &+ \frac{1}{2\times 3!}(E^{(+)}_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3}(1+\Gamma^{\sharp}) + E^{(-)}_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3}(1-\Gamma^{\sharp})) \\ &+ \frac{1}{5!}(I^{(+)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}(1+\Gamma^{\sharp}) + I^{(-)}_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}(1-\Gamma^{\sharp})). && (3.98) \end{aligned}$$

Later, we further decompose m_o as $m_o = m_o^{(+)} + m_o^{(-)}$ according to the $(\pm)$, namely the plus or minus chirality, in the above decomposition. The fermions are divided according to its chirality. The action is then written as follows.

$$\begin{aligned} S &= R(tr(m_e^2\Gamma^{\sharp}) - v^2 - 2i\bar{\phi}_R\psi_L) + tr(\frac{1}{3}m_e^3 + m_e m_o^2) \\ &- i(\bar{\phi}_R(m_e + v)\psi_L + \bar{\phi}_L(m_e+v)\psi_R + \bar{\phi}_L m_o\psi_L + \bar{\phi}_R m_o \psi_R) - \frac{1}{3}v^3. \end{aligned} \tag{3.99}$$

We integrate out the fields of order $\mathcal{O}(R)$ and consider the effective theory. In order to cope with the cubic term $tr(m_e^3)$, we consider the following rescaling:

$$\begin{aligned} & m_t = R\Gamma^{\sharp} + m = R\Gamma^{\sharp} + R^{-\frac{1}{2}}m'_e + R^{\frac{1}{4}}m'_o, \quad v_t = R + v = R + R^{-\frac{1}{2}}v', \\ & \psi = \psi_L + \psi_R = R^{-\frac{1}{2}}\psi'_L + R^{\frac{1}{4}}\psi'_R, \quad \bar{\phi} = \bar{\phi}_L + \bar{\phi}_R = R^{\frac{1}{4}}\bar{\phi}'_L + R^{-\frac{1}{2}}\bar{\phi}'_R. \end{aligned} \tag{3.100}$$

[8]This classical solution is impossible in the original $u(1|16,16)$ gauged theory, because the (33,33) component v is restricted to be an anti-hermitian matrix. If we are to consider the Wigner-Inönü contraction, one way is to consider the quintic action $S_{u(1|16,16)} = \frac{1}{5}Str(M_t^5) - R^4 Str M_t$. Then, the classical solution $\langle M \rangle = \left(\begin{array}{cc} R\Gamma^{\sharp}\otimes \mathbf{1}_{N\times N} & 0 \\ 0 & iR\otimes\mathbf{1}_{N\times N} \end{array}\right)$ is possible because $iR\otimes \mathbf{1}_{N\times N}$ is now an anti-hermitian matrix. Another caution is that the gauge group must be not $u(1|16,16)\otimes su(N,C)$ but $u(1|16,16)\otimes u(N,C)$. If the gauge group is $u(1|16,16)\otimes su(N,C)$, the linear term in (3.92) vanishes because the generators are $Tr(T^a)=0$, and the Wigner Inönü contraction is impossible from the beginning.

Following this rescaling, $tr(m_e'^3)$ and $\bar{\phi}'_R(m'_e+v')\psi_L$ are excluded, because this is rescaled as $\mathcal{O}(R^{-\frac{3}{2}})$. This theory is thus rescaled to be

$$S=(tr(m_e'^2\Gamma^\sharp)-v'^2+tr(m'_e m_o'^2))-i(2\bar{\phi}'_R\psi'_L+\bar{\phi}'_L(m'_e+v')\psi'_R+\bar{\psi}'_L m'_o\psi'_L+\bar{\phi}'_R m'_o\psi'_R). \quad (3.101)$$

We integrate out the fields m'_e, ψ'_L and $\bar{\phi}'_R$ by Gaussian integration. Completing this action square, we obtain

$$\begin{aligned} S &= tr(\{m'_e+\frac{1}{2}(m_o'^2\Gamma^\sharp+i(\psi'_R\bar{\phi}'_L)\Gamma^\sharp)\}^2\Gamma^\sharp)-\frac{1}{4}tr(\{m_o'^2+i(\psi'_R\bar{\phi}'_L)\}^2\Gamma^\sharp) \\ &- (v'+\frac{i}{2}(\bar{\phi}_L\psi_R))^2-\frac{1}{4}(\bar{\phi}'_L\psi_R)^2-2i(\bar{\phi}'_R+\frac{1}{2}\bar{\phi}'_L m_o)(\psi'_L+\frac{1}{2}m'_o\psi'_R)+\frac{i}{2}(\bar{\phi}'_L m_o'^2\psi'_R), \quad (3.102)\end{aligned}$$

where we have utilized the fact that $(\Gamma^\sharp)^2=\mathbf{1}_{32\times 32}$, and thus $\Gamma^\sharp$ possesses an inverse matrix. The effective action is given by the following path integral[9]. However, this effective action turns out to vanish:

$$\begin{aligned} & e^{-W}=\int dm'_e d\psi'_L d\bar{\phi}'_R dv e^{-S} \\ \Rightarrow\quad & W=-\frac{1}{4}tr(\{m_o'^2+i(\psi'_R\bar{\phi}'_L)\}^2\Gamma^\sharp)-\frac{1}{4}(\bar{\phi}'_L\psi_R)^2+\frac{i}{2}(\bar{\phi}'_L m_o'^2\psi'_R) \quad (3.103) \\ & =-\frac{1}{4}tr(m_o'^4\Gamma^\sharp)+\frac{i}{2}(tr(-m_o'^2(\psi'_R\bar{\phi}'_L)\Gamma^\sharp)+\bar{\phi}'_L m_o'^2\psi'_R)+\frac{1}{4}(tr((\psi'_R\bar{\phi}'_L)^2\Gamma^\sharp)-(\bar{\phi}'_L\psi'_R)^2)=0.\end{aligned}$$

That the first term vanishes is discerned from the anti-commutativity of the matrices m'_o and $\Gamma^\sharp$; namely $m'_o\Gamma^\sharp=-\Gamma^\sharp m'_o$. Noting this fact, we rewrite this term as

$$-\frac{1}{4}tr(m_o'^4\Gamma^\sharp)\overset{\{\Gamma_\sharp,m'_o\}=0}{=}\frac{1}{4}tr(m_o'^3\Gamma^\sharp m'_o)\overset{\text{cyclic}}{=}\frac{1}{4}tr(m_o'^4\Gamma^\sharp)=0. \quad (3.104)$$

While the effective action in the ten dimensions turns out to vanish, we investigate its symmetry. This model is invariant under the $gl(1|32,R)$ rotation $\delta M=[A,M]$ where $A=\begin{pmatrix} a & \chi \\ i\bar{\epsilon} & b\end{pmatrix}$. Like the $osp(1|32,R)$ nongauged model, the fermionic part of this rotation serves as the supercharge. In this sense, the $gl(1|32,R)$ supermatrix model also has $32+32=64$ supercharges. We consider the following rescaling of this parameter:

$$A=\begin{pmatrix} a'_e+R^{-\frac{3}{4}}a'_o & \chi'_L+R^{-\frac{3}{4}}\chi'_R \\ i(R^{-\frac{3}{4}}\bar{\epsilon}'_L+\bar{\epsilon}'_R) & b'\end{pmatrix}. \quad (3.105)$$

For the $gl(1|32,R)$ rotation

$$\delta M=[A,M]=\begin{pmatrix} [a,m+R\Gamma^\sharp]+i(\chi\bar{\phi}-\psi\bar{\epsilon}) & -(m+R\Gamma^\sharp)\chi+a\psi-b\psi+\chi v \\ i\bar{\epsilon}(m+R\Gamma^\sharp)-i(\bar{\phi}a+v\bar{\epsilon})+ib\bar{\phi} & i(\bar{\epsilon}\psi-\bar{\phi}\chi)+[b,v]\end{pmatrix}, \quad (3.106)$$

each component is transformed as follows:

$$\begin{aligned} \delta m'_e &= [a_e,m'_e]+[a_o,m'_o]+i(\chi'_L\bar{\epsilon}'_R+\chi'_R\bar{\epsilon}'_L)-i(\psi'_L\bar{\epsilon}'_R+\chi'_R\bar{\epsilon}'_L), && (3.107)\\ \delta m'_o &= \underline{[a_o,\Gamma^\sharp]}+[a'_e,m'_o]+i(\chi'_L\bar{\phi}'_L-\psi'_R\bar{\epsilon}'_R), && (3.108)\\ \delta\psi'_L &= -(m'_o\chi'_R+m'_e\chi_L)+(a_o\psi'_R+a_e\psi'_L)-b'\psi'_L+v'\chi'_L, && (3.109)\\ \delta\psi'_R &= \underline{2\chi'_R}+(a'_e\psi'_R-b\psi'_R-(m'_o\chi'_L)), && (3.110)\\ \delta\bar{\phi}'_L &= \underline{-2\bar{\epsilon}'_L}+((\bar{\epsilon}'_R m'_o)+b'\bar{\phi}'_L-\bar{\phi}'_L a'_e), && (3.111)\\ \delta\bar{\phi}'_R &= (\bar{\epsilon}'_R m'_e+\bar{\epsilon}'_L m'_o)+b'\bar{\phi}'_R-(\bar{\phi}'_L a'_o+\bar{\phi}'_R a'_e)-v'\bar{\epsilon}'_R, && (3.112)\\ \delta v' &= i(\bar{\epsilon}'_R\psi'_L+\bar{\epsilon}'_L m'_o)-i(\bar{\phi}'_R\chi'_L+\bar{\phi}'_L\chi'_R)+[b',v']. && (3.113)\end{aligned}$$

[9]The analogy of this path integral is a following toy model. (i)For the action $S=ax^2+bx+c$, the path integral is $e^{-W}=\int dx\exp(-(ax^2+bx+c))=\int dx\exp(-a(x+\frac{b}{2a})^2)\exp(-c+\frac{b^2}{4a})\propto\exp(-c+\frac{b^2}{4a})$. And thus the effective action is $W=\frac{-b^2+4ac}{4a}$. (ii)The second example is the following action, $S=axy+bx+cy$, with x,y being auxiliary fields. This integration is performed by $e^{-W}=\int dxdy\exp(a(x+\frac{b}{a})(y+\frac{c}{a})+\frac{bc}{a})$, and thus the effective action is $W=-\frac{bc}{a}$. This holds even if x,y are Grassmann odd quantities.

These results possess two major significances. First is that this reconfirms that the effective action (3.103) vanish. The effective action is invariant under the transformation of the fields m'_o, ψ'_R and $\bar{\phi}'_L$. Therefore, the effective action makes no difference if we translate these fermions arbitrarily. Taking the fields m'_o, ψ'_R and $\bar{\phi}'_L$ to be all zero, we find that the effective action is $W = 0$. Therefore, the effective action remains $W = 0$ even if we translate these fields into non-zero values.

The second significance is that this clarifies the structure of the supersymmetry of the effective theory. This gauged theory originally possesses no trivial translation like the $osp(1|32, R)$ non-gauged model, because this theory is deprived of the symmetry of the commutator. However, the Wigner-Inönü contraction serves to retrieve the translation in the ten dimensions. Actually, the underlined transformation of (3.108), (3.110) and (3.111) represents a translation independent of the matter field M. We thus find it natural to extract the structure of the IIB matrix model from the fields $m'^{(\pm)}_o$, ψ'_R and ϕ_L. Their supersymmetry transformation is read off from (3.108), (3.110) and (3.111) as

$$\delta m'^{(+)}_o = -i\psi'_R\bar{\epsilon}'_R, \quad \delta m'^{(-)}_o = +i\chi'_L\bar{\phi}'_L, \quad \delta\psi'_R = 2\chi'_R - (m'_o\chi'_L), \quad \delta\bar{\phi}'_L = -2\bar{\epsilon}'_L + (\bar{\epsilon}'_R m'_o). \tag{3.114}$$

This immediately gives the correspondence of the chirality among the bosonic fields and the fermionic fields: $(m^{(-)'}_o, \phi'_L)$ and $(m^{(+)'}_o, \psi'_R)$. This pairing also resembles the case of the $osp(1|32, R)$ nongauged model.

We first scrutinize the former correspondence. In this case, the homogeneous and the inhomogeneous supersymmetry is identified as

$$\delta^{(1)}_{\chi_L} = [Q_{\chi_L}, \bullet\,] = [\begin{pmatrix} 0 & \chi_L \\ 0 & 0 \end{pmatrix}, \bullet\,], \quad \delta^{(2)}_{\epsilon_L} = [Q_{\epsilon_L}, \bullet\,] = [\begin{pmatrix} 0 & 0 \\ i\bar{\epsilon}_L & 0 \end{pmatrix}, \bullet\,]. \tag{3.115}$$

Then, the rank-1 fields are transformed by this homogeneous transformation as

$$\delta^{(1)}_{\chi_L} A'^{(-)}_\mu = \frac{1}{32}tr(i\chi_L\bar{\phi}'_L\Gamma_\mu) - \frac{-1}{32}tr(i\chi_L\bar{\phi}'_L\Gamma_{i\sharp}) = -\frac{i}{16}\bar{\phi}'_L\Gamma_\mu\chi_L. \tag{3.116}$$

The inhomogeneous translation of course do not affect the transformation of the vector fields.

We next explore the commutation relations. Firstly, we discern that the commutator $[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\rho_L}]$, as well as $[\delta^{(2)}_{\epsilon_L}, \delta^{(2)}_{\eta_L}]$, manifestly vanish.

$$[\delta^{(1)}_{\chi_L}, \delta^{(1)}_{\rho_L}] = [[\begin{pmatrix} 0 & \chi_L \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \rho_L \\ 0 & 0 \end{pmatrix}], \bullet\,] = 0, \tag{3.117}$$

$$[\delta^{(2)}_{\epsilon_L}, \delta^{(2)}_{\eta_L}] = [[\begin{pmatrix} 0 & 0 \\ i\bar{\epsilon}_L & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ i\bar{\eta}_L & 0 \end{pmatrix}], \bullet\,] = 0, \tag{3.118}$$

This is a great advantage compared with the $osp(1|32, R)$ case, in which we are beset by the nonvanishing commutator of the homogeneous supersymmetry. The commutation relation $[\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_L}]$ is obtained by

$$[\delta^{(1)}_{\chi_L}, \delta^{(2)}_{\epsilon_L}]A'^{(-)}_\mu = \frac{iR}{16}\bar{\epsilon}_L\Gamma_\mu\chi_L. \tag{3.119}$$

The correspondence of the latter case goes in totally the same way. The supercharges are defined by

$$\delta^{(1)}_{\epsilon_R} = [Q_{\epsilon_R}, \bullet\,] = [\begin{pmatrix} 0 & 0 \\ i\bar{\epsilon}_R & 0 \end{pmatrix}, \bullet\,], \quad \delta^{(2)}_{\chi_R} = [Q_{\chi_R}, \bullet\,] = [\begin{pmatrix} 0 & \chi_R \\ 0 & 0 \end{pmatrix}, \bullet\,]. \tag{3.120}$$

Then, the supersymmetry transformation of the rank-1 fields is obtained by

$$\delta^{(1)}_{\epsilon_R} A'^{(+)}_\mu = \frac{1}{32}tr(-i\psi'_R\bar{\epsilon}_R\Gamma_\mu) + \frac{-1}{32}tr(-i\psi'_R\bar{\epsilon}_R\Gamma_{i\sharp}) = \frac{i}{16}\bar{\epsilon}_R\Gamma_\mu\psi'_R. \tag{3.121}$$

The commutators $[\delta^{(1)}_{\epsilon_R}, \delta^{(1)}_{\eta_R}]$ and $[\delta^{(2)}_{\chi_R}, \delta^{(2)}_{\rho_R}]$ vanish likewise, and the commutation relation $[\delta^{(1)}_{\epsilon_R}, \delta^{(2)}_{\chi_R}]$ gives

$$[\delta^{(1)}_{\epsilon_R}, \delta^{(2)}_{\chi_R}]A'^{(+)}_\mu = \frac{iR}{16}\bar{\epsilon}_R\Gamma_\mu\chi_R. \tag{3.122}$$

We see a better correspondence of the supersymmetry with the IIB matrix model than the $osp(1|32, R)$ model. However, the supersymmetry transformation of the fermion is not balanced by the commutator

$[A_\mu, A_\nu]$. This obstacles are shared with the $osp(1|32, R)$ case. The decisive drawback of this argument is that the ten-dimensional effective action totally vanishes. This comes from too large a symmetry of the model. However, it may be related to the notion of the topological matrix model [8], in which the IIB matrix model is induced ex nihilo. The exploration of such a gauged matrix model are interesting from various points of view, especially the existence of the local Lorentz invariance, and it is worth further investigations.

3.4 Curved-space classical solutions of the massive $osp(1|32, R)$ nongauged supermatrix model

In this section, we review the work [46], in which we elaborated on the relation between the supermatrix model and the curved-space background. In Section 2.4, we have introduced several generalizations of the IIB matrix model to accommodate the curved-space background. The work [46] focuses on the similarity between the massive IIB matrix model (2.140) and the $osp(1|32, R)$ supermatrix model with the quadratic term. This has led us to expect that the $osp(1|32, R)$ supermatrix model could also have interesting noncommutative static solutions. We delegate the detailed properties of the higher-dimensional fuzzy sphere, which plays a main role in this analysis, to Section 2.4.2.

3.4.1 Action

Here, we go back to the $osp(1|32, R)$ nongauged supermatrix model instead of the gauged matrix model, since our aim is to unravel the similarity with the alteration of the IIB matrix model described in (2.140). We start with the following massive model with the cubic interaction[10]:

$$\begin{aligned} S &= 3\mu Tr \sum_{Q=1}^{33} \left(\left(\sum_{p=1}^{32} M_p^Q M_Q^p \right) - M_{33}^Q M_Q^{33} \right) - iTr \sum_{Q,R=1}^{33} \left(\left(\sum_{p=1}^{32} M_P^Q [M_Q^R, M_R^P] \right) - M_{33}^Q [M_Q^R, M_R^{33}] \right) \\ &= 3\mu Tr(m_p^q m_q^p - 2i\bar{\psi}\psi) - iTr(m_p^q [m_q^r, m_r^p] - 3i\bar{\psi}_p [m_p^q, \psi_q]). \end{aligned} \tag{3.123}$$

Here, $P, Q, R, \cdots$ and $p, q, r, \cdots$ likewise run over $1, 2, \cdots, 33$ and $1, 2, \cdots, 32$. Here, we focus on the nongauged action, and this action has the separated $osp(1|32, R)$ rotational symmetry and the $U(N)$ gauge symmetry. We reduce the above matrix model to ten dimensions, by specializing the tenth space direction $x^{10} = x^\sharp$. In Section 3.2, we focused on the identification of the supersymmetry with that of the IIB matrix model, and we resorted to the chiral decomposition. However, we now focus on the nontrivial curved-space background, which leads us to reduce the bosonic 32×32 matrix m to the ten dimensions without the chiral decomposition. Namely, m is reduced as

$$m = W\Gamma^\sharp + A_\mu \Gamma^\mu + B_\mu \Gamma^{\mu\sharp} + \frac{1}{2!} C_{\mu_1\mu_2} \Gamma^{\mu_1\mu_2} + \frac{1}{4!} H_{\mu_1\cdots\mu_4} \Gamma^{\mu_1\cdots\mu_4\sharp} + \frac{1}{5!} Z_{\mu_1\cdots\mu_5} \Gamma^{\mu_1\cdots\mu_5}. \tag{3.124}$$

Then, the relevant action (3.123) is rewritten as

$$\begin{aligned} S &= 96\mu Tr \left(W^2 + A_\mu A^\mu - B_\mu B^\mu - \frac{1}{2} C_{\mu_1\mu_2} C^{\mu_1\mu_2} + \frac{1}{4!} H_{\mu_1\cdots\mu_4} H^{\mu_1\cdots\mu_4} + \frac{1}{5!} Z_{\mu_1\cdots\mu_5} Z^{\mu_1\cdots\mu_5} - \frac{i}{16}\bar{\psi}\psi \right) \\ &- 32iTr \Bigg(-3C_{\mu_1\mu_2}[A^{\mu_1}, A^{\mu_2}] + 3C_{\mu_1\mu_2}[B^{\mu_1}, B^{\mu_2}] + 6W[A_\mu, B^\mu] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3}, C^{\mu_3\mu_1}] \\ &\qquad + \frac{1}{4} B_{\mu_1}[H_{\mu_2\cdots\mu_5}, Z^{\mu_1\cdots\mu_5}] - \frac{1}{8} C_{\mu_1\mu_2}(4[H^{\mu_1}{}_{\rho_1\rho_2\rho_3}, H^{\mu_2\rho_1\rho_2\rho_3}] + [Z^{\mu_1}{}_{\rho_1\cdots\rho_4}, Z^{\mu_2\rho_1\cdots\rho_4}]) \\ &\qquad + \frac{3}{(5!)^2} \epsilon^{\mu_1\cdots\mu_{10}\sharp}(-W[Z_{\mu_1\cdots\mu_5}, Z_{\mu_6\cdots\mu_{10}}] + 10A_{\mu_1}[H_{\mu_2\cdots\mu_5}, Z_{\mu_6\cdots\mu_{10}}]) \end{aligned}$$

[10] We warn the readers that the notation of this section is different from that of [46]. In this paper, we define the path integral of the action S as

$$Z = \int dA \cdots e^{-S_E},$$

where S_E is defined in the ten-dimensional Euclidean space by the Wick rotation of the components $A_0, \cdots$ and the gamma matrices Γ^0, Therefore, the action is different from that of [46] by the overall sign.

$$+\frac{200}{(5!)^3}\epsilon^{\mu_1\cdots\mu_{10}\sharp}\left(5H_{\mu_1\cdots\mu_4}[Z_{\mu_5\mu_6\mu_7}{}^{\rho\chi}, Z_{\mu_8\mu_9\mu_{10}\rho\chi}] + 10H_{\mu_1\cdots\mu_4}[H_{\mu_5\mu_6\mu_7}{}^{\rho}, H_{\mu_8\mu_9\mu_{10}\rho}]\right.$$
$$\left. +6H^{\rho\chi}{}_{\mu_1\mu_2}[Z_{\mu_3\mu_4\mu_5\rho\chi}, Z_{\mu_6\cdots\mu_{10}}]\right)\Bigg)$$
$$- \quad 3Tr\left(\bar{\psi}\Gamma^{\sharp}[W,\psi] + \bar{\psi}\Gamma^{\mu}[A_\mu,\psi] + \bar{\psi}\Gamma^{\mu\sharp}[B_\mu,\psi] + \frac{1}{2!}\bar{\psi}\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2},\psi] + \right.$$
$$\left. +\frac{1}{4!}\bar{\psi}\Gamma^{\mu_1\cdots\mu_4\sharp}[H_{\mu_1\cdots\mu_4},\psi] + \frac{1}{5!}\bar{\psi}\Gamma^{\mu_1\cdots\mu_5}[Z_{\mu_1\cdots\mu_5},\psi]\right). \tag{3.125}$$

In the purely cubic supermatrix model (without mass term, which has been studied in [19,23,26,27]), the rank-2 field $C_{\mu_1\mu_2}$ possesses a cubic interaction term but has no quadratic term. This has been a severe obstacle to the appearance of a Yang-Mills-like structure in the supermatrix model, because it has been impossible to identify $C_{\mu_1\mu_2}$ with the commutators of the rank-1 fields $[B_{\mu_1}, B_{\mu_2}]$ (or $[A_{\mu_1}, A_{\mu_2}]$). In the eleven-dimensional case, this difficulty has been overcome in [37] through the addition of a mass term, and we thus expect this model to contain the massive IIB matrix model, the bosonic part of which has been studied in [29] to investigate perturbation theory around noncommutative curved-space backgrounds.

3.4.2 Resolution of the equations of motion

We proceed to search for possible curved-space classical configurations solving the equations of motion that follow from the action (3.123). To get a clearer picture of the problem, we now set the fermions and the positive mass-squared bosonic fields to zero:

$$\psi = W = A_\mu = H_{\mu_1\cdots\mu_4} = Z_{\mu_1\cdots\mu_5} = 0. \tag{3.126}$$

Since their masses are positive (at least in the spatial directions, while the time-like direction of quantum fields is generally unphysical), (3.126) is a stable classical solution. Furthermore, we choose to identify the tachyonic ten-dimensional vector field B_μ, rather than the well-defined A_μ with the bosonic fields of the massive IIB matrix model, in order to obtain a possibly stable curved-space classical solution. The classical equations of motion for the remaining tachyonic fields B_μ and $C_{\mu\nu}$ following from (3.123) are

$$B_\mu = -i\mu^{-1}[B^\nu, C_{\mu\nu}], \tag{3.127}$$
$$C_{\mu\nu} = -i\mu^{-1}([B_\mu, B_\nu] + [C_\mu{}^\rho, C_{\nu\rho}]). \tag{3.128}$$

Although it is difficult to solve these equations in full generality, the equation of motion for $C_{\mu\nu}$ suggests to take $C_{\mu\nu} \propto [B_\mu, B_\nu]$ for B_μ's satisfying a fairly simple commutator algebra. If we look for objects having a clear geometrical interpretation, it is tempting to look for solutions building fuzzy spheres.

$S^2 \times S^2 \times S^2$ classical solution

The simplest tentative solution is the product of three fuzzy 2-spheres with the symmetry $SO(3) \times SO(3) \times SO(3)$. Such a system is described by $N \times N$ hermitian matrices building a representation of the $so(3)(\sim su(2))$ Lie algebra in the following way:

$$[B_i, B_j] = i\mu r\epsilon_{ijk}B_k, \qquad B_1^2 + B_2^2 + B_3^2 = \mu^2 r^2\frac{N^2-1}{4}\mathbf{1}_{N\times N} \text{ for } (i,j,k = 1,2,3) \tag{3.129}$$

with similar relations for $i,j,k = 4,5,6$ and $i,j,k = 7,8,9$, trivial commutators for indices that do not belong to the same group of 3, and $B_0 = 0$ (ϵ_{ijk} is defined as usually).

This set of fields (3.129) describes a space formed by the Cartesian product of three fuzzy spheres located in the directions (x_1, x_2, x_3), (x_4, x_5, x_6) and (x_7, x_8, x_9), whose radii are all $\mu r\sqrt{N^2-1}/2$. $(N^2 - 1)/4$ is the quadratic Casimir operator of the $so(3)$ Lie algebra. Note that any positive-integer value of N is possible here, since N indexes the dimensions of irreducible representations. For $SO(3)$, the irreps have dimensions $N = 2j+1$, for all integer values of the spin j. However, we can also use spinorial representations with half-integer spins in this case. We have to consider this classical solution instead of the single S^2 fuzzy sphere

$$[B_i, B_j] = i\mu r\epsilon_{ijk}B_k \text{ (for } i,j,k = 1,2,3), \quad B_\mu = 0 \text{ (for } \mu = 0,4,5,\cdots,9), \tag{3.130}$$

because the solution $B_4 = \cdots = B_9 = 0$ is unstable in the directions 4 to 9 due to the negative mass-squared[11] of the rank-1 fields B_μ. Without restricting the generality, we can focus on the first sphere located in the direction (x_1, x_2, x_3), since the three fuzzy spheres all share the same equations of motion.

In the framework of fuzzy 2-spheres, we can solve the equations of motion (3.127) and (3.128) with the following ansatz for the rank-2 field C_{ij}:

$$C_{ij} = f(r)\epsilon_{ijk}B_k, \tag{3.131}$$

where $f(r)$ is a function depending on the radius parameter r. Indeed, the equation of motion (3.128) reduces then to:

$$\epsilon_{ijk}B_k(-f(r) + r + rf^2(r)) = 0. \tag{3.132}$$

(3.132) has two solutions: $f_\pm(r) = \frac{1\pm\sqrt{1-4r^2}}{2r}$. When we plug this result in the equation of motion for B (3.127), this leads to

$$B_i(1 - 2rf_\pm(r)) = 0. \tag{3.133}$$

This gives the same condition on the radius parameter r for both $f_+(r)$ and $f_-(r)$, namely:

$$\sqrt{1-4r^2} = 0. \tag{3.134}$$

Therefore, when we assume the ansatz (3.131), we obtain the classical solution (3.129) with the radius parameter set to $r = \frac{1}{2}$, which is fortunately real. Indeed, $r^2 \leq 0$ would indicate that the fuzzy sphere solution is unstable. For example, in the IIB massive matrix model described by (2.140), the sign of the squared radius of the fuzzy 2-sphere is linked to the sign of the mass term in the action and it would become negative for a correct-sign mass term, which is to be expected, since in that case, the trivial commutative solution becomes the stable vacuum of the theory.

We next want to discuss the stability of the $S^2 \times S^2 \times S^2$ classical solution in more qualitative terms [25]. To this end, we compare the energy of the trivial commutative solution $B_\mu = 0$ with that of the fuzzy-sphere solution. The classical energy for $B_\mu = 0$ is obviously $E_{B_\mu=0} = S_{B_\mu=0} = 0$.[12]

In the $S^2 \times S^2 \times S^2$ fuzzy-sphere background, the 2-form field C_{ij} is

$$C_{ij} = \epsilon_{ijk}B_k. \tag{3.135}$$

Therefore, the total energy is

$$\begin{aligned} E_{S^2} &= S_{S^2} = -64\mu\sum_{\mu=1}^{9}Tr(B_\mu B^\mu) = -3\times 64\mu\sum_{i=1}^{3}Tr(B_iB_i) \\ &= -12\mu^3N(N-1)(N+1). \end{aligned} \tag{3.136}$$

This result shows that the $S^2 \times S^2 \times S^2$ fuzzy-sphere classical solution has a lower energy compared to the trivial commutative solution and hence a higher probability.

Other curved-space solutions and the fuzzy 8-sphere

So far, we have considered the simplest curved-space solution $S^2 \times S^2 \times S^2$. Here, we consider the other curved-space solutions. We focus on the fuzzy 8-sphere solution, which has the $SO(9)$ rotational symmetry and is spanned by the nine-dimensional space. We assume the following fuzzy sphere solution

$$B_0^{(8)} = 0, \;\; B_p^{(8)} = \frac{\mu r}{2}G_p^{(8)}, \tag{3.137}$$

where p runs over the space component $1, 2, \cdots, 9$, and $G_p^{(8)}$ is already defined by (2.113). We likewise assume the ansatz for the rank-2 field $C_{\mu\nu}$ as

$$C_{0p}^{(8)} = 0, \;\; C_{pq}^{(8)} = -i\mu^{-1}g(r)[B_p^{(8)}, B_q^{(8)}]. \tag{3.138}$$

[11] The classical solution with $B_0 = 0$ has no problem, because it has a positive mass unlike the other directions of the field B.

[12] Since we now consider a classical solution with $B_0 = 0$ (thus no need of Wick rotation), the energy is simply equal to the classical action in which we substitute the solution.

Then, the equation of motion (3.128) implies

$$\frac{-i}{\mu}[B_p^{(8)}, B_q^{(8)}](-g(r) + 1 + 7r^2 g^2(r)) = 0. \tag{3.139}$$

We again have two choices for the function $f(r)$:

$$g_{\pm}(r) = \frac{1 \pm \sqrt{1 - 28r^2}}{14r^2}. \tag{3.140}$$

The equation of motion (3.127) for the rank-1 field $B_p^{(8)}$ gives

$$B_p^{(8)}(1 - 8r^2 g_{\pm}(r)) = 0. \tag{3.141}$$

Now, unlike the case of the $S^2 \times S^2 \times S^2$ fuzzy sphere, $1 - 8r^2 g_-(r) = 0$ does not have any real positive solution for r. However, there is exactly one such solution for $1 - 8r^2 g_+(r) = 0$, which is $r = \frac{1}{8}$.

More generally, for an S^{2k} fuzzy sphere $B_p^{(2k)} = (\frac{\mu r}{2})G_p^{(2k)}$, the same ansatz would give

$$\begin{aligned} g_{\pm}(r) &= \frac{1 \pm \sqrt{1 - 4(2k-1)r^2}}{2(2k-1)r^2}, \\ 1 - 2kr^2 g_{\pm}(r) &= 0, \text{ solvable only for } g_+(r) \text{ at } r = \frac{1}{2k}. \end{aligned}$$

We discuss the stability of the S^8 fuzzy-sphere classical solution by computing its classical energy. At the classical level, we obtain

$$E_{S^8} = -\frac{5}{8}\mu^3 n(n+8)N_4. \tag{3.142}$$

N_4 is given by (2.121), and we recall that this is given by $N_4 = (n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)/302400$. In contrast with the $S^2 \times S^2 \times S^2$ case, N can take here only certain precise values. For example, the smallest non-trivial representation ($n = 1$) has dimension 16, the following one ($n = 2$) 126, then 672, etc... The classical energy for the $S^2 \times S^2 \times S^2$ fuzzy-sphere solution is of the order $\mathcal{O}(-\mu^3 n^3) = \mathcal{O}(-\mu^3 N^3)$ while that of the fuzzy 8-sphere solution is of the order $\mathcal{O}(-\mu^3 n^{12}) = \mathcal{O}(-\mu^3 N^{\frac{6}{5}})$. Therefore, at large N, the $S^2 \times S^2 \times S^2$ triple fuzzy-sphere solution is energetically favored compared to the S^8 solution at the equal size N of the matrices. The presence of a spherical solution for all N in the $S^2 \times S^2 \times S^2$ case may indeed be a stabilizing factor. On the other hand, at equal value of n, whose physical meaning is less clear, the fuzzy 8-sphere solution has lower energy.

We have seen that the fuzzy 8-sphere is energetically favored. This is in contrast to the matrix model with the Chern-Simons term (2.111), in which the classical energy is given by (2.139) and thus the fuzzy 8-sphere is positive. In this case, the fuzzy 8-sphere is less energetically favored than the trivial commutative background. In this sense, this situation has more similarity to the IIB matrix model with the tachyonic mass term (2.140), in which the higher-dimensional fuzzy sphere, as well as the S^2 fuzzy sphere, is more energetically favored, as we have seen in (2.144).

The single fuzzy q-spheres for $q \leq 7$ do not constitute a stable classical solution of our model. When the S^q sphere occupies the direction $x_1, x_2, \cdots, x_{q+1}$, the solution $B_{q+2}^{(q)} = B_{q+3}^{(q)} = \cdots = B_9^{(q)} = 0$ is trivially unstable because of the negative mass-squared. Whereas, the Cartesian product of several fuzzy spheres, such as $S^2 \times S^5$, is a possible candidate for a stable classical solution.

3.4.3 Nucleation process of spherical branes

Starting from a vacuous spacetime, it is interesting to try to guess how spherical brane configurations could be successively produced through a sequence of decays into energetically more favorable meta-stable brane systems. The reader may have noticed that we have so far limited ourselves to the study of curved branes building irreducible representations of their symmetry groups. This could seem at first to be an unjustified prejudice, but it turns out that such configurations are energetically favored at equal values of N. For example, for $SO(3)$, an irreducible representation $\mathcal{R}_N$ of dimension N contributes as

$$E_{\mathcal{R}_N} = -4\mu^3(N^3 - N) \tag{3.143}$$

per fuzzy 2-sphere, while a reducible representation $\mathcal{R}_{N_1}\oplus\ldots\oplus\mathcal{R}_{N_m}$ of equal dimension $N_1+\ldots+N_m=N$ would contribute as

$$E_{\mathcal{R}_{N_1}\oplus\ldots\oplus\mathcal{R}_{N_m}} = -4\mu^3\sum_{i=1}^{m}(N_i^3-N_i). \tag{3.144}$$

This is obviously a less negative number, especially for big values of N. A similar conclusion was reached in [25] for the case of a Euclidean three-dimensional IIB matrix model with a Chern-Simons term and it seems to be a fairly general feature of matrix models admitting non-trivial classical solutions. This property is particularly clear for low-dimensional branes, since the classical energy is of order $\mathcal{O}(-\mu^3N^3)$ for the S^2 fuzzy sphere, but it remains true for any fuzzy $2k$-sphere solution, whose energy is of order $\mathcal{O}(-\mu^3N^{1+4/(k(k+1))})$, which also shows that low-dimensional configurations are favored. As hinted for in the preceding subsection, this latter fact can be physically understood by remarking that there are more irreps available for low-dimensional fuzzy spheres, which makes it easier for them to grow in radius through energetically favorable configurations. A third obvious fact is that configurations described by representations of high dimensionality are preferred.

Put together, these comparisons give us a possible picture for the brane nucleation process in this and similar matrix models. As they appear, configurations of all spacetime dimensions described by small representations will be progressively absorbed by bigger representations to form irreducible ones, that will slowly grow in this way to bigger values of N. Parallel to that, branes of higher dimensionalities will tend to decay into a bunch of branes of smaller dimensionalities, finally leaving only 2-spheres and noncommutative tori of growing radii. If the size of the hermitian matrices is left open, as is usually the case in completely reduced models, where the path integration contains a sum on that size, no configuration will be truly stable, since the size of the irreps will grow continuously.

Of course, this is a relatively qualitative study, which could only be proven correct by a full quantum statistical study of the model. However, it seems to be an interesting proposal for the possible physics of such theories.

3.4.4 Supersymmetry

We next comment on the structure of the supersymmetry. The biggest difference with the purely cubic supermatrix model, due to the addition of the mass term, is that this model is not invariant under the inhomogeneous supersymmetry

$$\delta_{\text{inhomogeneous}}m = 0, \quad \delta_{\text{inhomogeneous}}\psi = \xi, \tag{3.145}$$

which is a translation of the fermionic field. However, this model has 2 homogeneous supersymmetries in ten dimensions, which are part of the $osp(1|32,R)$ symmetry:

$$\delta_\epsilon M = \left[\begin{pmatrix} 0 & \epsilon \\ i\bar\epsilon & 0\end{pmatrix}, \begin{pmatrix} m & \psi \\ i\bar\psi & 0\end{pmatrix}\right] = \begin{pmatrix} i(\epsilon\bar\psi-\psi\bar\epsilon) & -m\epsilon \\ i\bar\epsilon m & 0\end{pmatrix}, \tag{3.146}$$

which transforms the bosonic and fermionic fields as

$$\delta_\epsilon m = i(\epsilon\bar\psi-\psi\bar\epsilon), \quad \delta_\epsilon\psi = -m\epsilon. \tag{3.147}$$

In the IIB matrix model, the supersymmetry has to balance between a quartic term $Tr([A_\mu,A_\nu])^2$ and a trilinear contribution $Tr\bar\psi\Gamma^\mu[A_\mu,\psi]$ in the action (2.1), which implies that the supersymmetry transformation of the fermionic field has to be bilinear in the bosonic field. On the other hand, the homogeneous supersymmetries are all linear in the fields in the purely cubic supermatrix model [26,27]. By incorporating the mass term, we are allowed to integrate out the rank-2 field $C_{\mu_1\mu_2}$ by solving the classical equation of motion iteratively as in [37][13].

[13] The Yang-Mills-like structure of the homogeneous supersymmetry transformation on the fermion comes from $-\frac{1}{2}C_{\mu\nu}\Gamma^{\mu\nu}\epsilon$, which is a part of $\delta_\epsilon\psi=-m\epsilon$. The explicit form of the iterative solution of the equations of motion (3.127) and (3.128) is

$$\begin{aligned} C_{\mu\nu} &= -i\mu^{-1}[B_\mu,B_\nu]+i\mu^{-3}[[B_\mu,B_\rho],[B_\nu,B^\rho]] \\ &\quad -i\mu^{-5}[[B_\mu,B_\rho],[[B_\nu,B_\chi],[B^\rho,B^\chi]]]+i\mu^{-5}[[B_\nu,B_\rho],[[B_\mu,B_\chi],[B^\rho,B^\chi]]]+\mathcal{O}(\mu^{-7}).\end{aligned}$$

Thanks to this procedure, the homogeneous SUSY transformation for the fermionic field becomes

$$\delta_\epsilon \psi = \frac{i}{2}[B_{\mu_1}, B_{\mu_2}]\Gamma^{\mu_1\mu_2}\epsilon + \cdots, \tag{3.148}$$

while the transformation of the field B_μ is

$$\delta_\epsilon B_\mu = -\frac{1}{32}tr_{32\times 32}(i(\epsilon\bar{\psi} - \psi\bar{\epsilon})\Gamma_{\mu\sharp}) = -\frac{i}{16}\bar{\epsilon}\Gamma_{\mu\sharp}\psi. \tag{3.149}$$

In that sense, the mass term is essential to realize the Yang-Mills-like structure for the homogeneous supersymmetries. On the other hand, if we want to preserve the homogeneous supersymmetries, we cannot just put a mass term for $C_{\mu_1\mu_2}$ by hand. The $osp(1|32, R)$ symmetry forces all fields to share the same mass, since they all lie in the same multiplet. In particular, we are forced to introduce a mass term for the fermions as well, which breaks the inhomogeneous supersymmetries. In other words, it seems difficult to have super-Yang-Mills-type structure for both homogeneous and inhomogeneous supersymmetries in the context of supermatrix models.

Indeed, in contrast with the purely cubic supermatrix model [26, 27], which has twice as many supersymmetry parameters, the massive supermatrix model has only $\mathcal{N} = 2$ supersymmetries in ten dimensions, because it lacks the inhomogeneous supersymmetries. In consequence, we cannot realize the translation of the vector field A_μ as a commutator of two linear combinations of the homogeneous and inhomogeneous supersymmetries (3.147) as in the IIB matrix model, where it leads to the interpretation of the eigenvalues of A_μ as spacetime coordinates. On the contrary,

$$[\delta_\epsilon, \delta_\chi]m = i[(\epsilon\bar{\chi} - \chi\bar{\epsilon}), m], \quad [\delta_\epsilon, \delta_\chi]\psi = i(\epsilon\bar{\chi} - \chi\bar{\epsilon})\psi \tag{3.150}$$

vanishes up to an $sp(32, R)$ rotation. This problem is a serious obstacle for the identification of the supersymmetry of this model with that of the IIB matrix model. More analysis will be reported elsewhere.

3.5 Other related studies

We have so far gone over the author's works [26, 46] concerning the supermatrix models. We conclude this section by touching on other related studies. In [37], Bagnoud, Carlevaro and Bilal discussed the $osp(1|32, R)$ supermatrix model in the twelve-dimensional and the eleven-dimensional contexts. Especially, they indicated how to perform the infinite momentum frame limit and used the matrix version of the T-duality to obtain a supersymmetric matrix quantum mechanics. They introduced the quadratic terms as in Sec. 3.4, and resorted to the perturbative technique that led to an infinite tower of the higher-order interaction among the physical fields. On the other hand, the lower-order term reproduced the BFSS model with an additional mass term together with interaction terms involving the five-brane degrees of freedom.

Other attempts for a background-independent matrix models have been achieved by the cubic action of the exceptional Jordan algebra. Firstly, L. Smolin proposed a matrix model based on the exceptional Jordan algebra whose automorphism group is F_4 (in this sense, we refer to this model as "the F_4 model" here). The exceptional Jordan algebra is related to the octonion, and gives a theory that has a compactification which reduces to the matrix string theory [6]. Its complex extension has been proposed in [35]. The E_6 exceptional Lie algebra is known to be the automorphism of the complexification. Therefore, the E_6 matrix model has twice as many degrees of freedom as the F_4 model, and derived the effective action of the matrix string type.

4 Matrix model with manifest general coordinate invariance

In the pervious section, we have investigated various aspects of the supermatrix models based on the $osp(1|32, R)$ super Lie algebra. This extension of the IIB matrix model poses a lot of interesting questions. Especially, in Sec. 3.3, we have investigated the gauged $gl(1|32, R) \otimes gl(R)$ model as a candidate for the extension of the IIB matrix model equipped with the local Lorentz invariance. The key idea is to render the parameter of the $SO(9, 1)$ Lorentz transformation dependent on the $u(N)$ matrices in a nontrivial manner. This is an essential alteration from the IIB matrix model, since the spacetime is interpreted as being immersed in the eigenvalues of the bosonic matrices A_μ. In addition, the $u(1|16, 16)$ (and its

analytic connection $gl(1|32,R)$) supermatrix model incorporates the higher-rank tensor field. Especially, the rank-3 field has a possibility to be identified with the spin connection in the quantum field theory on the curved spacetime.

In this section, we review the author's work [38] that inherits the above idea. The most serious obstacle of the $gl(1|32,R)$ supermatrix model is that the effective action reduced to the ten dimensions by the Wigner-Inönü contraction finally vanishes. This is ascribed to the fact that the element of $gl(1|32,R)$ of all ranks are allocated for both the matter fields and the parameter of the local Lorentz transformation. These two elements had to be mixed in the eleven-dimensional theory, due to the self-duality condition of the gamma matrices (A.11). Namely, even when the rank-6 fields come in the local Lorentz parameter due to the rule (3.67), this is identified with the rank-5 fields due to the self-duality rule (A.11). By the same token, the rank-7 matter fields coming from (3.66) are tantamount to the rank-4 fields.

In [38], we address the above approach in a different angle without the supermatrix model. We start from the ten-dimensional action from the outset, without the reduction from the eleven-dimensional model. Since we start from the ten-dimensional field theory, we can separate the matter fields and the local Lorentz parameter. We allocate the odd-rank fields for the matter fields, and the even-rank fields go for the local Lorentz parameter. In addition, we explicitly give an expansion of the matrices with the differential operators, to embody the idea of the identification of the matrices and the differential operators. In these senses, the work [38] gives a clearly different angle from our previous work in Sec. 3.3.

4.1 Bosonic part of the matrix model

This time, we start from the following action, defined in the ten-dimensional spacetime from the outset:

$$S = Tr_{N\times N}[tr_{32\times 32}V(m^2)]. \tag{4.1}$$

The uppercase Tr and the lowercase tr denote the trace for the $N\times N$ and 32×32 matrices, respectively. The indices $\mu,\nu,\cdots$ and $i,j,\cdots$ both run over $0,1,\cdots,9$.The former is allocated for the flat Minkowskian spacetime, and the latter is for the curved spacetime. $V(x)$ is some function determined later, and m consists of the odd-rank fields:

$$m = m_\mu\Gamma^\mu + \frac{i}{3!}m_{\mu_1\mu_2\mu_3}\Gamma^{\mu_1\mu_2\mu_3} - \frac{1}{5!}m_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5} - \frac{i}{7!}m_{\mu_1\cdots\mu_7}\Gamma^{\mu_1\cdots\mu_7} + \frac{1}{9!}m_{\mu_1\cdots\mu_9}\Gamma^{\mu_1\cdots\mu_9}. \tag{4.2}$$

Here, $m_{\mu_1\cdots\mu_{2n-1}}$ $(n=1,2,3,4,5)$ are all hermitian matrices, and this satisfies $\Gamma^0 m^\dagger\Gamma^0 = m$. In this sense, our action is shown to be hermitian as

$$S^\dagger = Tr[trV((m^\dagger)^2)] = Tr[trV((\Gamma^0 m\Gamma^0)^2)] = Tr[trV(m^2)] = S. \tag{4.3}$$

Here, we exclude the odd-power of m in the action, because m^{2k+1} all consists of the odd-rank gamma matrices and thus couple the fermions of the different chirality.

4.1.1 Identification of large N matrices with differential operators

We identify the space of the large-N matrices with that of the differential operators. By this identification, we can describe the differential operators on an arbitrary spin bundle over an arbitrary manifold in the continuum limit simultaneously, because they are embedded in the space of large N matrices, as is depicted in Fig. 8.

To illustrate this idea, we visit several simple examples on the differential operators of the scalar fields on two different bundle over the S_1 circle (see Fig. 9). We first consider the trivial bundle with the periodic condition $f(1)=f(0)$. We discritize the region $0\le x\le 1$ into small slices of spacing $\epsilon = \frac{1}{N}$. Then, the differential operator is approximated by the finite difference as

$$\partial_x f\left(\frac{k}{N}\right) \to \frac{1}{2}\left(\frac{f(\frac{k+1}{N}) - f(\frac{k}{N})}{\epsilon} + \frac{f(\frac{k}{N}) - f(\frac{k-1}{N})}{\epsilon}\right) = \frac{N}{2}\left(f\left(\frac{k+1}{N}\right) - f\left(\frac{k-1}{N}\right)\right). \tag{4.4}$$

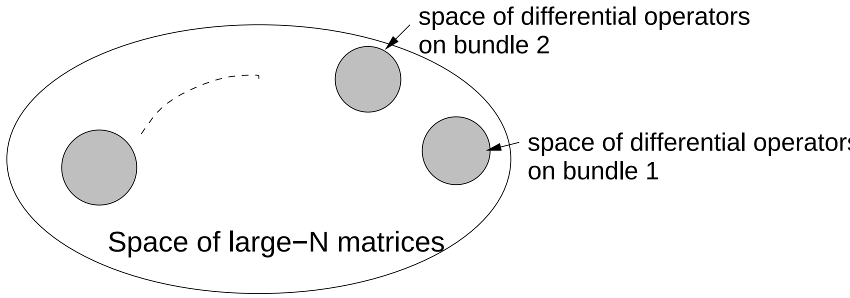

Figure 8: The way the spaces of the differential operators are embedded in the space of large N matrices.

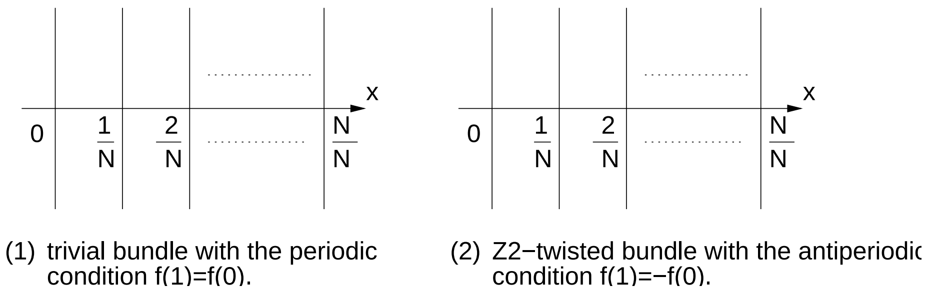

Figure 9: Differential operators on (1)the trivial bundle and (2)the Z_2-twisted bundle over S_1.

Due to the periodic condition, this finite difference is expressed by the large N matrix as

$$\partial_x \to A = \frac{N}{2}\begin{pmatrix} 0 & 1 & & & & -1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & & \ddots & & \\ 1 & & & & -1 & 0 \end{pmatrix}. \tag{4.5}$$

We next consider the similar problem with respect to the Z_2-twisted bundle, in which the antiperiodic condition $f(1) = -f(0)$ is imposed. Paying attention to this antiperiodicity, we understand that the finite difference in the discritized space is expressed as

$$\partial_x \to A = \frac{N}{2}\begin{pmatrix} 0 & 1 & & & & 1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & & \ddots & & \\ -1 & & & & -1 & 0 \end{pmatrix}. \tag{4.6}$$

In the following, from the big space of large N matrices, we pick up a subspace consisting of the differential operators over one manifold. We regard m as the differential operators over this manifold. Then, we analyze the effective theory of the fields appearing in the expansion of the differential operators (the explicit form is given later).

The space of the differential operator is infinite-dimensional. The trace Tr for this space is generally divergent. However, we can render this trace finite by choosing the function $V(m^2)$ damping rapidly. We would like to identify this matrix m with the Dirac operator in the curved spacetime. Clearly, the Dirac operator has a dimensionality $[(\text{length})^{-1}]$, while m is a dimensionless quantity. Then, we introduce a constant τ with the dimensionality $[(\text{length})^2]$, and express m as

$$\begin{aligned} m &= \tau^{\frac{1}{2}} D, \text{ where} \\ D &= A_\mu \Gamma^\mu + \frac{i}{3!} A_{\mu_1\mu_2\mu_3}\Gamma_{\mu_1\mu_2\mu_3} - \frac{1}{5!}A_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5} - \frac{i}{7!}A_{\mu_1\cdots\mu_7}\Gamma^{\mu_1\cdots\mu_7} + \frac{1}{9!}A_{\mu_1\cdots\mu_9}\Gamma^{\mu_1\cdots\mu_9}. \end{aligned} \tag{4.7}$$

Here, D and $A_{\mu_1\cdots\mu_{2n-1}}$ are matrices of dimensionality $[(\text{length})^{-1}]$. τ is similar to the Regge slope α' in string theory, and is introduced as a reference scale. This parameter τ is not a cut-off parameter. $V(m^2)$ is an exponentially decreasing function, and the damping factor is supplied by the action itself. Since $V(m^2)$ is a function of the dimensionless quantity m, τ represents a damping scale. When we approximate the differential operators by finite N matrices, an N-dependent ultraviolet cut-off naturally appears. When we take N to infinity, this cut-off becomes infinitely small. But the scale τ is completely independent of this ultraviolet cut-off, and takes a constant value even in the large N limit. Thus, we can fairly take N to infinity and identify the large N matrices with the differential operators, at least when we investigate the effective theory at tree level. In this sense, our model *differs* from the induced gravity.

The matrices $A_{\mu_1\cdots\mu_{2n-1}}$ are expanded by the number of the derivatives. The hermiticity of $A_{\mu_1\cdots\mu_{2n-1}}$ leads us to expand it by the anti-commutator of the derivatives as

$$A_{\mu_1\cdots\mu_{2n-1}} = a_{\mu_1\cdots\mu_{2n-1}}(x) + \sum_{k=1}^{\infty}\frac{i^k}{2}\{\partial_{i_1}\partial_{i_2}\cdots\partial_{i_k}, a^{(i_1\cdots i_k)}{}_{\mu_1\cdots\mu_{2n-1}}(x)\}. \tag{4.8}$$

Here, the indices $i_1,\cdots,i_k$ are symmetric, while the indices $\mu_1,\cdots,\mu_{2k-1}$ are anti-symmetric. Since we identify D with the extension of the Dirac operator in the curved spacetime, we find it natural to identify the coefficients $a^{(i)}_{\mu}(x)$ with the vielbein of the background metric. A simple dimensional analysis immediately indicates that the coefficients $a^{(i_1\cdots i_k)}_{\mu_1\cdots\mu_{2n-1}}(x)$ has a dimensionality $[(\text{length})^{-1+k}]$. Then, D is identified with the Dirac operator in the curved space as

$$D = e^{\frac{1}{2}}(x)\left[ie_{\mu}{}^{i}(x)\Gamma^{\mu}\left(\partial_i + \frac{1}{4}\Gamma^{\nu\rho}\omega_{i\nu\rho}(x)\right)\right]e^{-\frac{1}{2}}(x) + (\text{higher-rank and higher-derivative terms}). \tag{4.9}$$

4.1.2 Local Lorentz invariance

We next clarify that our matrix model is invariant under the local Lorentz transformation. Here, the local Lorentz transformation is given by

$$\begin{aligned}
\delta m &= [m,\varepsilon], \text{ where} \\
\varepsilon &= -i\varepsilon_{\emptyset} + \frac{1}{2!}\Gamma^{\mu_1\mu_2}\varepsilon_{\mu_1\mu_2} + \frac{i}{4!}\Gamma^{\mu_1\cdots\mu_4}\varepsilon_{\mu_1\cdots\mu_4} - \frac{1}{6!}\Gamma^{\mu_1\cdots\mu_6}\varepsilon_{\mu_1\cdots\mu_6} \\
&- \frac{i}{8!}\Gamma^{\mu_1\cdots\mu_8}\varepsilon_{\mu_1\cdots\mu_8} + \frac{1}{10!}\Gamma^{\mu_1\cdots\mu_{10}}\varepsilon_{\mu_1\cdots\mu_{10}}.
\end{aligned} \tag{4.10}$$

ε is the parameter of the local Lorentz transformation, and it satisfies $\Gamma^0\varepsilon^{\dagger}\Gamma^0 = \varepsilon$. The coefficients $\varepsilon_{\emptyset},\cdots,\varepsilon_{\mu_1\cdots\mu_{10}}$ are all hermitian matrices. The discussion in (3.67) immediately indicates that the closure of the algebra of the local Lorentz transformation necessitates all the even-rank fields. The invariance of the action is verified as

$$\delta S = Tr[tr(2V'(m^2)m[m,\varepsilon])] = 0. \tag{4.11}$$

Here, $V'(x)$ denotes $V'(x) = \frac{\partial V(x)}{\partial x}$. The vanishing of (4.11) can be verified by noting the cyclic rule of the trace Tr. Since we are now discussing the space of the infinite-dimensional space, it is not necessarily trivial whether the trace satisfies the cyclic rule. However, the cyclic rule holds when the coefficients $a^{(i_1\cdots i_k)}{}_{\mu_1\cdots\mu_{2n-1}}(x)$ damps rapidly. The only nontrivial part for the cyclic rule is the commutator between the derivative and the fields

$$Tr([\partial_i, a^{(j_1\cdots j_k)}{}_{\mu_1\cdots\mu_{2n-1}}(x)]) = \int d^dx\langle x|(\partial_i a^{(j_1\cdots j_k)}{}_{\mu_1\cdots\mu_{2n-1}}(x))|x\rangle = \int d^dx(\partial_i a^{(j_1\cdots j_k)}{}_{\mu_1\cdots\mu_{2n-1}}(x))\langle x|x\rangle.$$

However, this surface term does not contribute when the coefficients $a^{(i_1\cdots i_k)}{}_{\mu_1\cdots\mu_{2n-1}}(x)$ damp rapidly.

4.1.3 Determination of $V(m^2)$

We next determine the function $V(m^2)$. We require that the action (4.1) should have the flat metric $m_0 = i\tau^{\frac{1}{2}}\Gamma^{\mu}\partial_{\mu}$ as a classical background. However, this turns out to be a draconian constraint on

$V(x)$. The bosonic part is expressed by the heat-kernel expansion. We delegate the details of the heat-kernel expansion to Appendix B. We find it more convenient to express $V(u)$ in terms of its Laplace transformation

$$V(u) = \int_0^\infty ds g(s) e^{-su}, \tag{4.12}$$

in order to see a more transparent correspondence with the heat kernel. Then, the action is now expressed as

$$Tr[trV(m^2)] = \int_0^\infty dsg(s)Tr[tre^{-s\tau D^2}] = \int \frac{d^d x}{(2\pi\tau)^{\frac{d}{2}}} \left(\sum_{k=-\infty}^{\infty} \left(\int_0^\infty dsg(s)s^{-\frac{d}{2}+k} \right) \tau^k \mathcal{A}_k(x) \right), \tag{4.13}$$

where $\mathcal{A}_k(x)$ are the Seeley-de-Witt coefficients of the heat kernel $Tr[tr(e^{-\tau D^2})]$, namely

$$Tr[tr(e^{-\tau D^2})] \quad = \quad \int \frac{d^d x}{(2\pi\tau)^{\frac{d}{2}}} \sum_{k=-\infty}^{\infty} \tau^k \mathcal{A}_k(x). \tag{4.14}$$

Here, k runs all the integers. We can show that the half-integer k can be excluded in the following way. The coefficients $\mathcal{A}_k(x)$ has a dimensionality [(length)$^{-2k}$]. On the other hand the terms $\prod_{k=1}^n (\partial_{i_1} \cdots \partial_{i_{p_k}} a^{(j_1 \cdots j_{l_k})}{}_{\mu_1 \cdots \mu_{2m_k-1}}(x))$ has a dimensionality $\mathcal{D} = -n + \sum_{i=1}^n (-p_i + l_i)$. Therefore, this belongs to the coefficients $\mathcal{A}_{-\frac{\mathcal{D}}{2}}(x)$. On the other hand, this term has $-n + \sum_{i=1}^n (2m_i + p_i + l_i)$ indices, which is equal to $\mathcal{D}$ modulo 2. Therefore, when $\mathcal{D}$ is odd, these terms cannot contract the indices, and thus cannot survive in the action. If such terms existed, they would no longer be a scalar and thus would violate the Lorentz invariance of the action. In this way, we eliminate the contribution of the half-integer k.

Then, let us think about the condition for m_0 to be a classical solution. To this end, the linear terms of all the fluctuations must vanish in the action. Now, only the scalar fields can contribute to the action, we have only to consider the linear terms $a_\mu{}^{(\mu i_1 i_1 i_2 i_2 \cdots i_l i_l)}(x)$ (for $l = 0, 1, 2, \cdots$). Since $a^{(i)}{}_\mu(x)$ is identified with the vielbein, its cancellation means the cancellation of the cosmological constant. This amounts to the cancellation of the Seeley-de-Witt coefficient $\mathcal{A}_0(x)$. We disregard the derivative terms $\partial_{i_1} \cdots a^{(j_1 \cdots)}{}_{\mu_1 \cdots \mu_{2n-1}}(x)$, because these can be eliminated by the partial integration. Since the fields $a_\mu{}^{(\mu i_1 i_1 i_2 i_2 \cdots i_l i_l)}(x)$ has a dimensionality [(length)^{2l}], it belongs to the Seeley-de-Witt coefficient $\mathcal{A}_{-l}(x)$. Therefore, we generally demand that the coefficients $\mathcal{A}_{-l}(x)$ for $l = 0, 1, 2, \cdots$ should all cancel.

However, it turns out that this imposes a demanding condition on the function $V(m^2)$. The above condition is translated as

$$\int_0^\infty dsg(s)s^{-\frac{d}{2}-n} = 0, \text{ for } n = 0, 1, 2, \cdots. \tag{4.15}$$

This condition can further be rewritten as

$$\int_0^\infty du V(u) u^{n-1+\frac{d}{2}} = \int_0^\infty du \int_0^\infty dsg(s)e^{-su}u^{n+\frac{d}{2}-1} = \Gamma(n + \frac{d}{2}) \int_0^\infty dsg(s)s^{-\frac{d}{2}-n} = 0, \tag{4.16}$$

for $n = 0, 1, 2, \cdots$. One example for such a function is the following:

$$V_0(u) = \frac{\partial^{\frac{d}{2}-1}(\exp(-u^{\frac{1}{4}})\sin(u^{\frac{1}{4}}))}{\partial u^{\frac{d}{2}-1}}. \tag{4.17}$$

We can show that the function (4.17) satisfies the condition (4.16) as follows [71]. Firstly, we note the following integral

$$\int_0^\infty dy y^m e^{-ay} = m! a^{-m-1}, \text{ where } a = \exp(\frac{i\pi}{4}) = \frac{1+i}{\sqrt{2}}. \tag{4.18}$$

This is a real number when $m - 3$ is a multiple of 4. Taking the imaginary part of the both hand sides, we obtain

$$\int_0^\infty dy y^{4n+3} \sin(\frac{y}{\sqrt{2}}) \exp(-\frac{y}{\sqrt{2}}) = 0, \text{ for } n = 0, 1, 2, \cdots. \tag{4.19}$$

We make a substitution $u = \frac{y^4}{4}$ and perform the partial integration to obtain the solution (4.17)[14].

We then derive the Einstein gravity in the low energy limit, from this bosonic part. The linear term of the vielbein ${a^{(i)}}_{\mu}(x)$ is eliminated, and thus the graviton is now massless since the mass term vanishes due to the general coordinate invariance. When we retain the curved-space Dirac operator in the expansion (4.9), the Seeley-de-Witt expansion is now obtained by

$$Tr[tr(e^{-\tau D^2})] = \int d^d x \frac{32}{(2\pi\tau)^{\frac{d}{2}}} e(x) \left(\tau \frac{R(x)}{6} + \cdots \right). \tag{4.20}$$

This term should be clearly allocated for the coefficient $\mathcal{A}_1(x)$.

We next give a qualitative argument for the mass and the kinetic terms of the matter fields. While the explicit heat-kernel expansion of (4.9) entails an involved calculation, it is easy to estimate which coefficient $\mathcal{A}_k(x)$ the mass term and the kinetic term belong to via the dimensional analysis. Since the coefficients ${a^{(i_1\cdots i_k)}}_{\mu_1\cdots\mu_{2n-1}}(x)$ and ${a^{(j_1\cdots j_l)}}_{\mu_1\cdots\mu_{2n-1}}(x)$ respectively have the dimensionality $[(\text{length})^{-1+k}]$ and $[(\text{length})^{-1+l}]$, we have

$$\text{mass terms:} {a^{(i_1\cdots i_k)}}_{\mu_1\cdots\mu_{2n-1}}(x) {a^{(j_1\cdots j_l)}}_{\mu_1\cdots\mu_{2n-1}}(x) \in \mathcal{A}_{1-\frac{k+l}{2}}(x), \tag{4.21}$$

$$\text{kinetic terms:} (\partial_{k_1} {a^{(i_1\cdots i_k)}}_{\mu_1\cdots\mu_{2n-1}}(x))(\partial_{k_2} {a^{(j_1\cdots j_l)}}_{\mu_1\cdots\mu_{2n-1}}(x)) \in \mathcal{A}_{2-\frac{k+l}{2}}(x). \tag{4.22}$$

Here, $k+l$ must be an even number. Firstly, the mass term and the kinetic terms of the fields $a_{\mu_1\cdots\mu_{2n-1}}(x)$ belong to $\mathcal{A}_1(x)$ and $\mathcal{A}_2(x)$, respectively. The explicit heat-kernel calculation reveals that the relevant mass terms are given by

$$\begin{aligned} Tr[tr(e^{-\tau D^2})] \;\; = \;\; & \int d^d x \frac{32}{(2\pi\tau)^{\frac{d}{2}}} \tau \left(\frac{2}{3!} a_{\mu_1\cdots\mu_3}(x) a^{\mu_1\cdots\mu_3}(x) + \frac{4}{5!} a_{\mu_1\cdots\mu_5}(x) a^{\mu_1\cdots\mu_5}(x) \right. \\ & \left. + \frac{6}{7!} a_{\mu_1\cdots\mu_7}(x) a^{\mu_1\cdots\mu_7}(x) + \frac{8}{9!} a_{\mu_1\cdots\mu_9}(x) a^{\mu_1\cdots\mu_9}(x) \right) + \cdots . \end{aligned} \tag{4.23}$$

We give the derivation of this result in Appendix. B. Thus, the fields $a_{\mu_1\cdots\mu_{2n-1}}(x)$ for $n = 2,3,4,5$ are clearly massive. While (4.23) cancels the mass term $a_\mu(x) a^\mu(x)$, the field $a_\mu(x)$ is also presumably massive because there is no reason to prohibit the cross terms $a_\mu(x) {a^{(i_1\cdots i_k)}}_{\mu}(x)$.

Next, we consider the fields ${a^{(i)}}_{a_1\cdots a_{2n-1}}(x)$. Especially the fields ${a^{(i)}}_{ia_1\cdots a_{2n}}(x)$ (with $n = 1,2,3,4$) are identified with the anti-symmetric tensor fields in the type IIB supergravity. Their mass terms are included in $\mathcal{A}_0(x)$, while the kinetic terms reside in $\mathcal{A}_1(x)$. These fields are therefore regarded as massless. This observation augurs well for the matrix model which reduces to the type IIB supergravity.

However, it is not clear whether the higher-spin fields ${a^{(i_1\cdots i_k)}}_{a_1\cdots a_{2n-1}}(x)$ ($k = 2,3,\cdots$) are massive, since the mass terms and the kinetic terms belong to the coefficients $\mathcal{A}_{1-k}(x)$ and $\mathcal{A}_{2-k}(x)$ respectively and both of them vanish in the action.

4.2 Supersymmetric action

In the previous section, we have discerned that the bosonic part of the matrix model reduces to the Einstein gravity in the low energy limit. Our goal is to establish the extension of the IIB matrix model that reduces to the type IIB supergravity in the low energy limit. This leads us to scrutinize the fermionic part. We pay attention of the correspondence of the supersymmetry.

We start from the following action:

$$S_S = Tr[tr(V(m^2))] + Tr\bar{\psi} m \psi. \tag{4.24}$$

Here, m is defined in the same way as in the bosonic model. ψ is a Weyl fermion. However, the crucial difference from the IIB matrix model is that we abstain its Majorana property. Namely, we abstain its

[14] This is the famous counter-example against Hausdorff's moment problem when we replace the finite integral with the infinite integral. When the function $V(u)$ satisfies

$$\int_0^1 du u^n V(u) = 0, \text{ for } n = 0,1,2,\cdots,$$

The function $V(u)$ must be zero constantly. This problem is called "Hausdorff's moment problem" [71]. However, this is not the case with (4.16) because we are considering the infinite integral.

hermiticity. It is extremely difficult to build a matrix model that closes with respect to the local Lorentz transformation $\delta\psi = \frac{1}{4}\Gamma^{\mu\nu}\{\varepsilon_{\mu\nu}, \psi\}$. That is why we abandon the hermiticity and define the local Lorentz transformation as $\delta\psi = \frac{1}{4}\Gamma^{\mu\nu}\varepsilon_{\mu\nu}\psi$. The fermionic field is also expanded by the number of the derivatives as

$$\psi = \left(\chi(x) + \sum_{l=1}^{\infty}\chi^{(i_1\cdots i_l)}(x)\partial_{i_1}\cdots\partial_{i_l}\right)(e^{-\tau D^2})^{\alpha}. \tag{4.25}$$

Clearly, the fields $\chi^{(i_1\cdots i_l)}(x)$ has the dimensionality $[(\text{length})^l]$. We put the damping factor $(e^{-\tau D^2})^{\alpha}$, so that the action $\bar{\psi}m\psi$ should be finite. We choose this power in accordance with the function (4.17), which leads us to set $\alpha = \frac{1}{4}$.

4.2.1 Local Lorentz invariance

We discuss its local Lorentz invariance, which is defined as

$$\delta m = [m, \varepsilon], \quad \delta\psi = \epsilon\psi, \tag{4.26}$$

where ε is defined in (4.10). $\bar{\psi}$ is transformed as

$$\delta\bar{\psi} = \delta(\psi^{\dagger}(\Gamma^0)) = -\psi^{\dagger}(\Gamma^0)^2\varepsilon^{\dagger}\Gamma^0 = -\bar{\psi}\epsilon. \tag{4.27}$$

The action is transformed by this local Lorentz transformation as

$$\delta S_S = 2Tr[tr(V_S'(m^2))[m,\varepsilon]] + Tr[tr(\bar{\psi}[m,\varepsilon]\psi)] = 0. \tag{4.28}$$

This holds true when the coefficients $a^{(i_1\cdots i_k)}{}_{\mu_1\cdots\mu_{2n-1}}(x)$ and $\chi^{(i_1\cdots i_k)}(x)$ damp rapidly in the infinite distance.

4.2.2 Determination of $V_S(m^2)$

We next discuss the determination of the function $V_S(m^2)$. Here, we do not give a specific form of $V_S(m^2)$ and we just list up the conditions for $V_S(m^2)$.

Firstly, the flat background $m_0 = i\tau^{\frac{1}{2}}\Gamma^{\mu}\partial_{\mu}$ must be a classical solution. As we have seen in the bosonic case, this imposes a strict condition on the function $V_S^2(m^2)$; namely this must satisfy the criterion (4.16).

Secondly, this model must retain the even-rank anti-symmetric tensor fields $a^{(i)}{}_{i\mu_1\cdots\mu_{2n}}(x)$ ($n = 1,2,3,4$). The action $V_S(m^2)$ must be determined so that these fields should be massless and thus be bequeathed in the low-energy limit.

It is not clear whether the function (4.17) satisfies the latter condition, since we have yet to elucidate the mass of the higher-spin fields.

4.2.3 Supersymmetry

We next discuss its supersymmetry. We define the supersymmetric transformation as

$$\delta_{\epsilon}m = \epsilon\bar{\psi} + \psi\bar{\epsilon}, \quad \delta_{\epsilon}\psi = 2V_S'(m^2)\epsilon. \tag{4.29}$$

Here, $V_S'(x) = \frac{\partial V_S(x)}{\partial x}$. Now, we assume that the function $V_S(u)$ can be expanded around $u = 0$ as

$$V_S(u) = \sum_{k=1}^{\infty}\frac{a_{2k}}{2k}u^k. \tag{4.30}$$

Therefore, the following argument does not apply to the function (4.17), because it is not possible to expand this around the origin $u = 0$. The supersymmetry transformation of $\bar{\psi}$ is derived as

$$\begin{aligned}\delta_{\epsilon}\bar{\psi} &= \delta_S(\psi^{\dagger}\Gamma^0) = -\epsilon^{\dagger}(\Gamma^0)^2(\sum_{k=0}^{\infty}a_{2(k+1)}m^{2k})^{\dagger}\Gamma^0 \\ &= -\epsilon^{\dagger}\Gamma^0(\sum_{k=0}^{\infty}(-1)^{2k-1}a_{2(k+1)}(\Gamma^0 m^{\dagger}\Gamma^0)^{2k})\Gamma^0 = 2\bar{\epsilon}V_S'(m^2).\end{aligned} \tag{4.31}$$

Then, the supersymmetry invariance of the action can be easily verified as

$$\delta_\epsilon S_S = Tr[tr(2V_S'(m^2)m(\epsilon\bar{\psi}+\psi\bar{\epsilon}) + 2\bar{\psi}mV_S'(m^2)\epsilon + 2\bar{\epsilon}mV_S'(m^2)\psi)] = 0. \tag{4.32}$$

We next analyse the commutation relation of the supersymmetry transformation. This is given by

$$[\delta_\epsilon, \delta_\xi]m = 2[(\xi\bar{\epsilon} - \epsilon\bar{\xi}), V_S'(m^2)], \tag{4.33}$$

$$[\delta_\epsilon, \delta_\xi]\psi = 2\psi\left(\bar{\epsilon}m\frac{V_S'(m^2) - V_S'(0)}{m^2}\xi - \bar{\xi}m\frac{V_S'(m^2) - V_S'(0)}{m^2}\epsilon\right), \tag{4.34}$$

with the help of the equation of motion

$$\frac{\partial S_S}{\partial\bar{\psi}} = 2m\psi = 0, \quad \frac{\partial S_S}{\partial\psi} = 2\bar{\psi}m = 0. \tag{4.35}$$

This can be verified by taking the difference of the two transformations, as we did for the IIB matrix model or the supermatrix model. For the boson, we compare the following two transformation:

$$\begin{aligned} m &\stackrel{\delta_\xi}{\to} m + \xi\bar{\psi} + \psi\bar{\xi} \stackrel{\delta_\epsilon}{\to} m + (\epsilon+\xi)\bar{\psi} + \psi(\bar{\epsilon}+\bar{\xi}) + 2\xi\bar{\epsilon}V_S'(m^2) + 2V_S'(m^2)\epsilon\bar{\xi}, \\ m &\stackrel{\delta_\epsilon}{\to} m + \epsilon\bar{\psi} + \psi\bar{\epsilon} \stackrel{\delta_\xi}{\to} m + (\epsilon+\xi)\bar{\psi} + \psi(\bar{\epsilon}+\bar{\xi}) + 2\epsilon\bar{\xi}V_S'(m^2) + 2V_S'(m^2)\xi\bar{\epsilon}. \end{aligned}$$

For the fermion, we compare the following two paths:

$$\begin{aligned} \psi &\stackrel{\delta_\xi}{\to} \psi + 2V_S'(m^2)\xi \\ &\stackrel{\delta_\epsilon}{\to} \psi + 2V_S'(m^2)(\epsilon+\xi) + \sum_{k=2}^{\infty} a_{2k}[(\epsilon\bar{\psi}+\psi\bar{\epsilon})m^{2k-3} + m(\epsilon\bar{\psi}+\psi\bar{\epsilon})m^{2k-4} + \cdots m^{2k-3}(\epsilon\bar{\psi}+\psi\bar{\epsilon})]\xi, \\ \psi &\stackrel{\delta_\epsilon}{\to} \psi + 2V_S'(m^2)\epsilon \\ &\stackrel{\delta_\xi}{\to} \psi + 2V_S'(m^2)(\epsilon+\xi) + \sum_{k=2}^{\infty} a_{2k}[(\xi\bar{\psi}+\psi\bar{\xi})m^{2k-3} + m(\xi\bar{\psi}+\psi\bar{\xi})m^{2k-4} + \cdots m^{2k-3}(\xi\bar{\psi}+\psi\bar{\xi})]\epsilon. \end{aligned}$$

In order to see the structure of the supersymmetry transformation of the $\mathcal{N}=2$ supersymmetry. Since we now abstain the hermiticity of the fermion, we separate the fermions into the real part and the imaginary part as

$$\epsilon = \epsilon_1 + i\epsilon_2, \quad \xi = \xi_1 + i\xi_2, \tag{4.36}$$

where $\epsilon_1, \epsilon_2, \xi_1$ and ξ_2 are Majorana-Weyl fermions. Now, we assume that the supersymmetry parameters are proportional to the unit matrix for brevity, namely

$$\epsilon_{1,2} = c_{\epsilon_{1,2}}\mathbf{1}_{N\times N}, \quad \xi_{1,2} = c_{\xi_{1,2}}\mathbf{1}_{N\times N}, \tag{4.37}$$

where $c_{\epsilon_{1,2}}$ and $c_{\xi_{1,2}}$ are real Grassmann-odd c-numbers. Another assumption is that the background metric is flat.

Firstly, the transformation of the bosonic field is calculated as

$$\begin{aligned} [\delta_\epsilon, \delta_\xi]A_\mu &= \frac{1}{16}tr([\delta_\epsilon,\delta_\xi]m\Gamma_\mu) = \frac{1}{16}\sum_{k=2}^{\infty} a_{2k}tr(\xi\bar{\epsilon}m^{2k-2}\Gamma_\mu - \epsilon\bar{\xi}m^{2k-2}\Gamma_\mu - m^{2k-2}\xi\bar{\epsilon}\Gamma_\mu + m^{2k-2}\epsilon\bar{\xi}\Gamma_\mu) \\ &= \frac{1}{16}\sum_{k=2}^{\infty} a_{2k}(\bar{\xi}[m^{2k-2},\Gamma_\mu]\epsilon - \bar{\epsilon}[m^{2k-2},\Gamma_\mu]\xi) \\ &= \frac{a_4}{16}(\bar{\xi}[\Gamma^{\nu_1}\Gamma^{\nu_2},\Gamma_\mu]\epsilon - \bar{\epsilon}[\Gamma^{\nu_1}\Gamma^{\nu_2},\Gamma_\mu]\xi)A_{\nu_1}A_{\nu_2} + \cdots \\ &= \frac{a_4}{16}(\bar{\xi}\Gamma^i\epsilon - \bar{\epsilon}\Gamma^i\xi)[A_i, A_\mu] + \cdots = \frac{a_4}{8}(\bar{\xi}_1\Gamma^i\epsilon_1 + \bar{\xi}_2\Gamma^i\epsilon_2)[A_i,A_\mu] + \cdots, \end{aligned} \tag{4.38}$$

We focus on the supersymmetry transformation of the vector field $a_\mu(x)$. The commutator $[A_i, A_\mu]$ represents the translation and the gauge transformation:

$$[A_i, A_\mu] = [i\partial_i + a_i(x), i\partial_\mu + a_\mu(x)] + \cdots = \underbrace{i(\partial_i a_\mu(x))}_{\text{translation}} \underbrace{-i(\partial_\mu a_i(x)) + [a_i(x), a_\mu(x)]}_{\text{gauge transformation}} + \cdots. \tag{4.39}$$

Therefore, the commutator of the bosonic fields represents the right translation of the vector fields. However, this does not hold true of the fermionic fields. The commutator is computed as

$$\begin{aligned}[][\delta_\epsilon, \delta_\xi]\psi &= -\sum_{k=2}^{n} a_{2k}\psi(\bar{\xi}m^{2k-3}\epsilon - \bar{\epsilon}m^{2k-3}\xi) + \cdots = -a_4(\bar{\xi}\Gamma^j\epsilon - \bar{\epsilon}\Gamma^j\xi)\psi A_j + \cdots \\ &= -2a_4(\bar{\xi}_1\Gamma^j\epsilon_1 + \bar{\xi}_2\Gamma^j\epsilon_2)\psi A_j + \cdots .\end{aligned}$$

We have a closer look at the term ψA_j:

$$\psi A_j = i\psi\partial_j + \cdots = \left(\chi(x)\partial_j + \sum_{i=1}^{\infty}\chi^{(i_1\cdots \imath_l)}(x)\partial_{i_1}\cdots\partial_{i_l}\partial_j\right)(e^{-\tau D^2})^\alpha + \cdots, \tag{4.40}$$

where $\cdots$ denotes the omission of the non-linear terms of the fields. Therefore, each fermionic field is subject to the following transformation:

$$[\psi_\epsilon, \psi_\xi]\chi(x) = 0 + \cdots, \;\; [\psi_\epsilon, \psi_\xi]\chi^{(i_1\cdots i_{l+1})}(x) = -2a_4(\bar{\xi}_1\Gamma^j\epsilon_1 + \bar{\xi}_2\Gamma^j\epsilon_2)\chi^{(i_1\cdots i_l}(x)\delta^{i_{l+1})j} + \cdots . \tag{4.41}$$

Therefore, the fermionic fields are not subject to the translation by the commutator of the supersymmetry.

However, the argument in this section is only applicable to the function $V'_S(u)$ that can be expanded around $u = 0$, and does not hold true of the function (4.17). We speculate that we may find a right supersymmetry for such functions.

We surmise that the fermionic fields $\chi(x)$ and $\chi^{(i)}(x)$, which are respectively identified with the dilatino and the gravitino, would be massless due to the supersymmetry, when the even-rank fields ${a^{(i)}}_{i\mu_1\cdots\mu_{2n}}(x)$ are massless. It is an intriguing future problem to seek such an action more rigorously.

5 Monte Carlo simulation of the fuzzy sphere solutions

We have seen several generalizations of the IIB matrix model to accommodate the curved-space classical background in Sec. 2.4. We have reviewed the author's work [46] in Sec. 3.4 about the curved-space background in the context of the supermatrix model.

In this section, we address the stability of the curved-space background in the quantum sense, based on the author's work [60]. It has been a conundrum to discuss the stability of the fuzzy sphere in the quantum sense. There have been hitherto several attempts for the quantum stability. In [25], the quantum stability has been discussed in terms of the one-loop effective action. In [51, 55, 59], they performed the two-loop perturbative calculation of the Feynman diagram, to unravel the stability of the fuzzy sphere and other backgrounds. Another attempt is to investigate the quantum stability via the Gaussian expansion, and the first-order calculation is given in [52].

Here, we address the quantum stability of the fuzzy sphere via the heat bath algorithm of the Monte Carlo simulation, whose detail we delegate to Appendix C. Our approach is totally different from these foregoing works [25, 51, 52, 55, 59], in the sense that we discuss the stability nonperturbatively via the numerical simulation. The Monte Carlo simulation has played a pivotal role in elucidating the nonperturbative aspects of the lattice gauge theory. By the same token, the application of the Monte Carlo simulation to the IIB matrix model has given us many interesting nonperturbative insights. In [13], the bosonic IIB matrix model has been scrutinized nonperturbatively via the Monte Carlo simulation. This analysis [13] was followed by the supersymmetric extension [20, 22] via the hybrid Monte Carlo simulation.

Here, we scrutinize the simplest case for the matrix model with the curved-space classical solution. Namely, we concentrate on the three-dimensional bosonic IIB matrix model with the Chern-Simons term which incorporates the S^2 fuzzy sphere solution:

$$S = NTr\left(-\frac{1}{4}\sum_{\mu,\nu=1}^{3}[A_\mu, A_\nu]^2 + \frac{2i\alpha}{3}\sum_{\mu,\nu,\rho=1}^{3}\epsilon_{\mu\nu\rho}A_\mu A_\nu A_\rho\right). \tag{5.1}$$

The notation is the same as that in Sec. 2.4.1, except we set $\frac{1}{g^2} = N$ in this section. We delegate the miscellaneous properties of the S^2 fuzzy sphere classical solution $A_\mu = \alpha L_\mu$ to Sec. 2.4.1.

5.1 Nonperturbative studies of the fuzzy sphere initial condition

We start with the Monte Carlo simulation for the initial condition of the irreducible representation of the $N \times N$ fuzzy sphere classical solution:

$$A_\mu^{(0)} = \alpha L_\mu. \tag{5.2}$$

Starting from this initial condition, we see whether or not the fuzzy sphere decays due to the quantum effect of the matrix model. Especially, the histogram of the eigenvalues of the Casimir operator

$$Q = A_1^2 + A_2^2 + A_3^2 \tag{5.3}$$

describes the fuzzy sphere's stability visually. In the following, x denotes the eigenvalues of Q, and $f(x)$ denotes the eigenvalue distribution function normalized as $\int_0^\infty dx f(x) = 1$. At the initial state, the histogram is peaked at $R^2 = \frac{N^2-1}{4}\alpha^2$, because the Casimir is proportional to the unit matrix for the fuzzy sphere. If the fuzzy sphere is stable in the quantum sense, the histogram does not decay greatly from the peaked distribution. On the other hand, if the fuzzy sphere is unstable, the histogram undergoes a drastic deformation.

5.1.1 Nonperturbative stability of the fuzzy sphere

We first investigate the $N = 16$ and $\alpha = 0.5, 1.0, 2.0$ case starting from the initial condition (5.2). We see a clear difference between the $\alpha = 0.5$ and the $\alpha = 1.0, 2.0$ case.

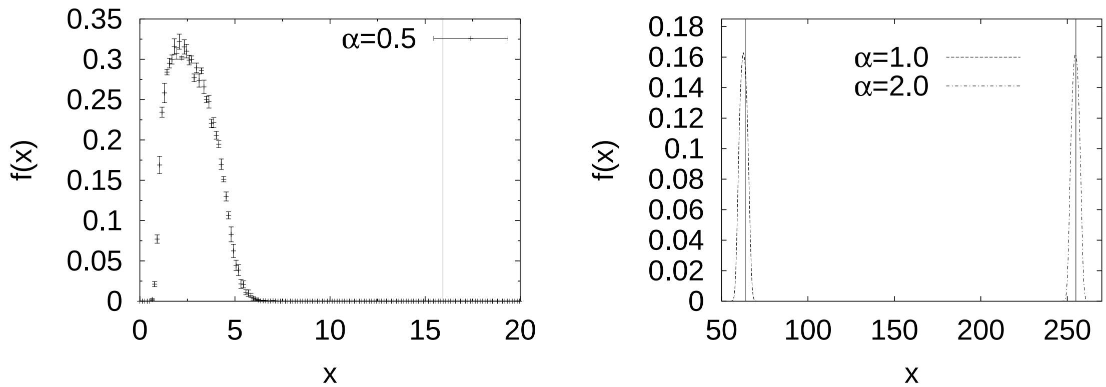

Figure 10: The instability at $\alpha = 0.5$(left) and the stability at $\alpha = 1.0, 2.0$(right) of the fuzzy sphere for $N = 16$.

We see from Fig. 10 the instability of the fuzzy sphere at $\alpha = 0.5$. The vertical lines in the histogram denote the radius-square of the fuzzy sphere classical solution $R^2 = \alpha^2 \frac{N^2-1}{4}$. Now, this value is 15.9375 for $\alpha = 0.5$, and the eigenvalues of Q were originally peaked there. This fuzzy sphere decays due to the quantum effect and the eigenvalues are concentrated far away from the original fuzzy sphere configuration.

However, this situation changes for $\alpha = 1.0, 2.0$. Fig. 10 indicates that the eigenvalue distribution constitutes only one lump in the vicinity of the classical value of the radius-square (which is 63.75 and 255 for $\alpha = 1.0$ and $\alpha = 2.0$ respectively), and that the eigenvalues are not dissipated otherwise at all. This implies that the eigenvalues remain constituting a thin sphere shell near its original fuzzy sphere configuration. In this sense, we regard the fuzzy sphere classical solution as stable. This stability is also reconfirmed in Section 5.3.1, by demonstrating the dynamical evolution of the single fuzzy sphere state from the initial condition $A_\mu^{(0)} = 0$.

5.1.2 First-order phase transition

In the previous section, we have discerned that the fuzzy sphere solution. In this section, we deepen the above observation and demonstrate the first-order transition of the matrix model (5.1).

Firstly, we ascribe the above stability of the fuzzy sphere to the meager quantum effect of the model. The effective action W of this matrix model is given by

$$W = -\log\left(\int dA e^{-S}\right). \tag{5.4}$$

The effect of the classical fuzzy sphere solution is of the order $\mathcal{O}(\alpha^4 N^4)$, as one can easily verify by plugging the fuzzy sphere solution (2.78) into the action (5.1). On the other hand, the effect of the path integral measure is of the order $\mathcal{O}(N^2)$. This comparison leads to the observation that the matrix model receives only a small quantum effect at

$$\alpha \gg \mathcal{O}(\frac{1}{\sqrt{N}}). \tag{5.5}$$

To elaborate on this observation, we next calculate the following quantities, for $N = 8, 16, 32$ with the fuzzy sphere initial condition (5.2).

1. The action $\langle S \rangle$: This quantity gives the energy of the numerical configuration.
2. The spacetime extent $\langle \frac{1}{N} Tr A_\mu^2 \rangle$: This provides us with the information of the eigenvalue distribution, and hence the stability of the fuzzy sphere.
3. The Yang-Mills term: $\langle \frac{1}{N} Tr F_{\mu\nu}^2 \rangle$: This quantity, as well as the spacetime extent, have been already investigated in the bosonic IIB matrix model without the Chern-Simons term (this amounts to the $\alpha = 0$ case) in [13]. For the bosonic IIB matrix model, both are shown to behave at the order $\mathcal{O}(1)$. It is interesting to compare the behavior of this quantity with that of the bosonic IIB matrix model.
4. We derive the following exact result from the Schwinger-Dyson equation of the matrix model:

   $$0 = \int dA \frac{\partial}{\partial A_\mu^a} (Tr(t^a A_\mu) e^{-S}). \tag{5.6}$$

   Here, t^a $(a = 1, 2, \cdots, N^2 - 1)$ are the basis of the $SU(N)$ Lie algebra, and the matrices A_μ are expanded as $A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a t^a$. The Schwinger-Dyson equation (5.6) gives

   $$\langle K \rangle = \frac{1}{N} \langle Tr(-[A_\mu, A_\nu]^2 + 2i\alpha \epsilon_{\mu\nu\rho} A_\mu A_\nu A_\rho) \rangle = 3(1 - \frac{1}{N^2}) = K_0, \tag{5.7}$$

   as we prove explicitly in Appendix C.4. While K is given analytically, the numerical computation of its vacuum expectation $\langle K \rangle$ is significant for the legitimacy of the algorithm.
5. The Chern-Simons term:

   $$\langle M \rangle = \langle \frac{2i}{3N} Tr \epsilon_{\mu\nu\rho} A_\mu A_\nu A_\rho \rangle. \tag{5.8}$$

   We have obtained the analytical result K, which amounts to

   $$K = \frac{1}{N} Tr F_{\mu\nu}^2 + 3\alpha M. \tag{5.9}$$

   While $\langle K \rangle$ itself obviously behaves as $\mathcal{O}(1)$, it is interesting to discern the behavior of each of the separate terms M, as well as $\frac{1}{N} Tr F_{\mu\nu}^2$.

We postpone the plot of $\langle K \rangle$ later, and plot the other four quantities. The above power-counting observation for the quantum effect motivates us to plot these quantities against $\tilde{\alpha} = \alpha\sqrt{N}$, instead of α. For the fuzzy sphere classical solution, they respectively scale as

$$S = \mathcal{O}(\alpha^4 N^4) = \mathcal{O}(\tilde{\alpha}^4 N^2), \;\; \frac{1}{N} Tr A_\mu^2 = \mathcal{O}(\alpha^2 N^2) = \mathcal{O}(\tilde{\alpha}^2 N),$$
$$\frac{1}{N} Tr F_{\mu\nu}^2 = \mathcal{O}(\alpha^4 N^2) = \mathcal{O}(\tilde{\alpha}^4), \;\; M = \mathcal{O}(\alpha^3 N^2) = \mathcal{O}(\tilde{\alpha}^3 \sqrt{N}). \tag{5.10}$$

This leads us to plot the quantities $\frac{1}{N^2}\langle S \rangle$, $\frac{1}{N}\langle \frac{1}{N} Tr A_\mu^2 \rangle$, $\langle \frac{1}{N} Tr F_{\mu\nu}^2 \rangle$ and $\frac{1}{\sqrt{N}}\langle M \rangle$ against $\tilde{\alpha}$ in Fig. 11.

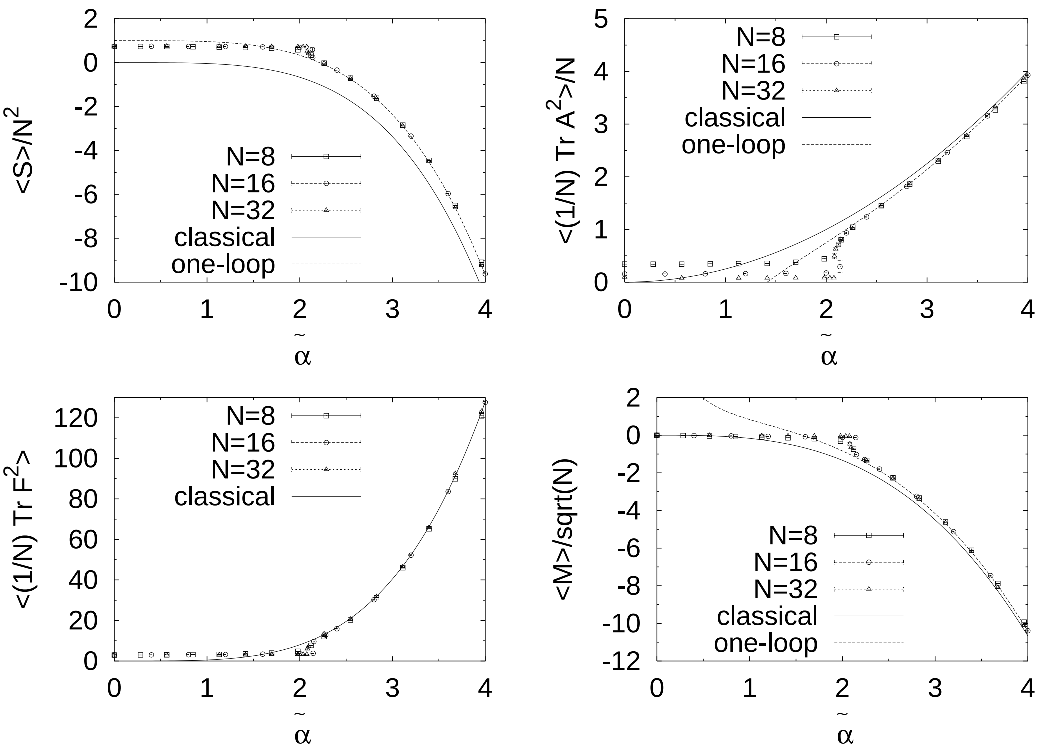

Figure 11: $\frac{1}{N^2}\langle S\rangle$ (upper left), $\frac{1}{N}\langle \frac{1}{N}TrA_\mu^2\rangle$ (upper right), $\langle \frac{1}{N}TrF_{\mu\nu}^2\rangle$ (lower left) and $\frac{1}{\sqrt{N}}\langle M\rangle$ (lower right) against $\tilde{\alpha}$, for $N = 8, 16, 32$ with the fuzzy sphere initial condition (5.2).

This plotting poses a serious consequence. This matrix model has the first-order phase transition with the change of the parameter $\tilde{\alpha}$. In the above four cases, we see a discontinuity at the critical point

$$\tilde{\alpha} = \tilde{\alpha}_{cr}^{(l)} \sim 2.1. \tag{5.11}$$

We also find that this discontinuity is steeper for the larger N. This indicates that there is a first-order phase transition between the two phases.

We call the phase at $\tilde{\alpha} < \tilde{\alpha}_{cr}^{(l)}$ the "Yang-Mills phase". In this phase, the quantum effect of the matrix model becomes strong, and thus the fuzzy sphere classical solution is no longer stable. Rather, the behavior for this phase has a similarity to that of the bosonic IIB matrix model without the Chern-Simons term. Fig. 11 clearly indicates that this model undergoes a smooth transition from $\alpha = 0$ (the bosonic IIB matrix model itself) to the small α. The vacuum expectation values of $\langle \frac{1}{N}TrA_\mu^2\rangle$ and $\langle \frac{1}{N}TrF_{\mu\nu}^2\rangle$ have been analyzed in [13], where they are shown to behave at $\mathcal{O}(1)$. This behavior is inherited in the Yang-Mills phase. We elaborate on this phase more closely in Section 5.2.1.

We call the phase at $\tilde{\alpha} > \tilde{\alpha}_{cr}^{(l)}$ the "fuzzy sphere phase", where the quantum effect is so small that the classical fuzzy sphere is stable. Now, we discern that the fuzzy sphere is stable at $N = 16$, $\alpha = 1.0, 2.0$, because the system is in the fuzzy sphere phase in this case. They are scaled as (5.10), and this behavior is totally different from that of the Yang-Mills phase.

This first-order phase translation is a product of the the Monte Carlo simulation, which could not be seen by the perturbative approaches. In this sense, we accentuate that this first-order phase transition is a nonperturbative phenomenon.

We next exhibit the corresponding plot of the quantity $\langle K\rangle$ in Fig. 12. In the Yang-Mills phase, the observables $\langle \frac{1}{N}TrF_{\mu\nu}^2\rangle$ and $\langle M\rangle$ behave as $\mathcal{O}(1)$, similarly to the bosonic IIB matrix model. The fluctuation of K is thus small in the Yang-Mills phase.

On the other hand, $\langle \frac{1}{N}TrF_{\mu\nu}^2\rangle$ and $3\alpha\langle M\rangle$ both behave as $\mathcal{O}(\tilde{\alpha}^4)$ in the fuzzy sphere phase. Nevertheless, the quantity $\langle K\rangle = \langle(\frac{1}{N}TrF_{\mu\nu}^2 + 3\alpha M)\rangle$ is of the order $\mathcal{O}(1)$, as is analytically verified. This is ascribed to the huge cancellation between these two terms in the fuzzy sphere phase. Therefore, the fluctuation of $\langle K\rangle$ is naturally bigger in the fuzzy sphere phase than in the bosonic Yang-Mills phase.

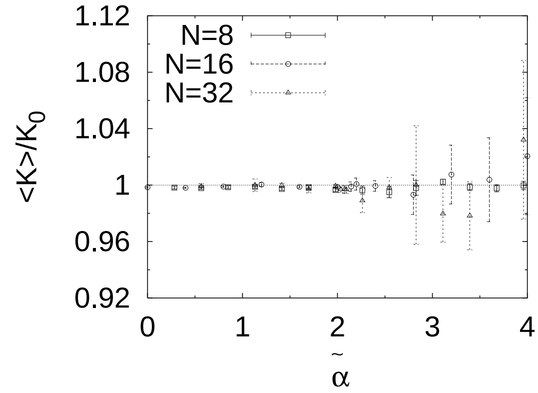

Figure 12: The plot of the quantity $\frac{\langle K\rangle}{K_0}$, for the fuzzy sphere initial condition (5.2).

5.1.3 One-loop dominance in the fuzzy sphere phase

The behavior of these four quantities in the fuzzy sphere phase gives an even more striking consequence. The numerical value of these four quantities coincides their one-loop results. The one-loop vacuum expectation values are calculated in the large-N limit as

$$\frac{\langle S\rangle}{N^2} = -\frac{1}{24}\tilde{\alpha}^4 + 1, \quad \frac{1}{N}\langle \frac{1}{N}TrA_\mu^2\rangle = \frac{\tilde{\alpha}^2}{4} - \frac{1}{\tilde{\alpha}^2}, \quad \langle \frac{1}{N}TrF_{\mu\nu}^2\rangle = \frac{\tilde{\alpha}^2}{2}, \quad \frac{1}{\sqrt{N}}\langle M\rangle = -\frac{\tilde{\alpha}^3}{6} + \frac{1}{\tilde{\alpha}}. \tag{5.12}$$

We delegate the derivation of these results to the appendices of [60]. The first term comes from the classical effect, while the one-loop correction comes in the second term of each quantity. The dotted lines in the graphs denote these one-loop result of these quantities[15]. The coincidence of the numerical values of these observables with the one-loop result suggests the higher-loop effects are suppressed in the large-N limit and hence that the one-loop effect is dominant in the matrix model (5.1).

5.1.4 Width of the eigenvalue distribution

We next discuss the dependence of the width of the lump in the histogram of the Casimir Q on the parameter α and N. We define the width σ as

$$\sigma^2 = \int_0^\infty dx x^2 f(x) - \left(\int_0^\infty dx x f(x)\right)^2 = \langle \frac{1}{N}Tr(A_\mu^2)^2\rangle - \langle \frac{1}{N}TrA_\mu^2\rangle^2 = \langle \frac{1}{N}\sum_{i=1}^N \lambda_i^2\rangle - \langle \frac{1}{N}\sum_{i=1}^N \lambda_i\rangle^2. \tag{5.13}$$

Here, $\lambda_1, \lambda_2, \cdots, \lambda_N$ are the eigenvalues of Q. The above three expressions are all tautological. For the convenience of comparing the histograms for different N and α, we plot the eigenvalue distribution against the $(x - x_0)$ in Fig. 13, instead of the usual argument x, where $x_0(N, \tilde{\alpha})$ are the mean of the observed eigenvalues for a particular N and $\tilde{\alpha}$ and are functions of N and $\tilde{\alpha}$.

The overlapping of the different α histograms for a particular N shows that the width of the eigenvalue distribution does not depend on α. We also observe that the width of the distributions for different N has a linear dependence on $\log N$ in Fig. 14. The width σ turns out to behave as

$$\sigma^2 \sim 2\log N - 0.84. \tag{5.14}$$

This is in agreement with the one-loop calculation $\sigma^2 = 2\log N$, which is also derived in the appendices of [60].

[15]For $\langle \frac{1}{N}TrF_{\mu\nu}^2\rangle$, we omit the dotted line, because it is not subject to the one-loop correction.

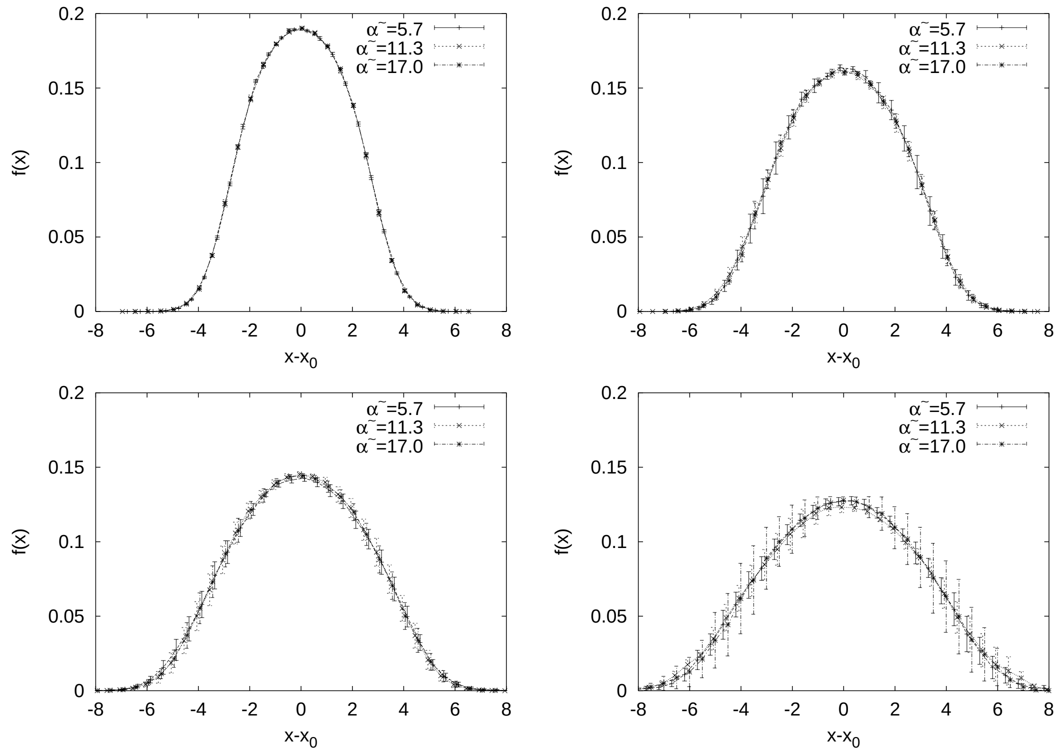

Figure 13: Eigenvalue distribution of Q around its mean, for $N = 8$ (upper left), $N = 16$ (upper right), $N = 32$ (lower left) and $N = 64$ (lower right), respectively.

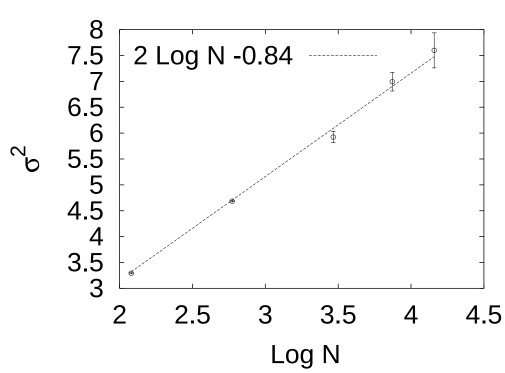

Figure 14: The square of the width of the distribution σ^2 against $\log N$.

5.2 Analysis of the Yang-Mills phase

We have so far investigated the numerical simulation starting from the fuzzy sphere initial condition (5.2). We have found that there is a phase transition between the Yang-Mills phase and the fuzzy sphere phase at $\tilde{\alpha}_{cr}^{(l)} \sim 2.1$ (namely, $\alpha_{cr}^{(l)} \sim \frac{2.1}{\sqrt{N}}$). In this section, we elaborate on the behavior of the Yang-Mills phase, and we launch the Monte Carlo simulation from the initial condition

$$A_{\mu}^{(0)} = 0. \tag{5.15}$$

5.2.1 Eigenvalue distribution in the Yang-Mills phase

We start with discussing the eigenvalue distribution of the Casimir Q, for $N = 8, 16, 32$ with $\alpha = 0.0$ to indicate the N dependence and $\alpha = 0.0, 0.2, 0.4, 0.6$ with $N = 32$ to indicate the α dependence. Here, we plot the eigenvalue density $\rho(r)$ against $r = \sqrt{(\text{eigenvalues})}$ in Fig. 15. This r trivially represents the

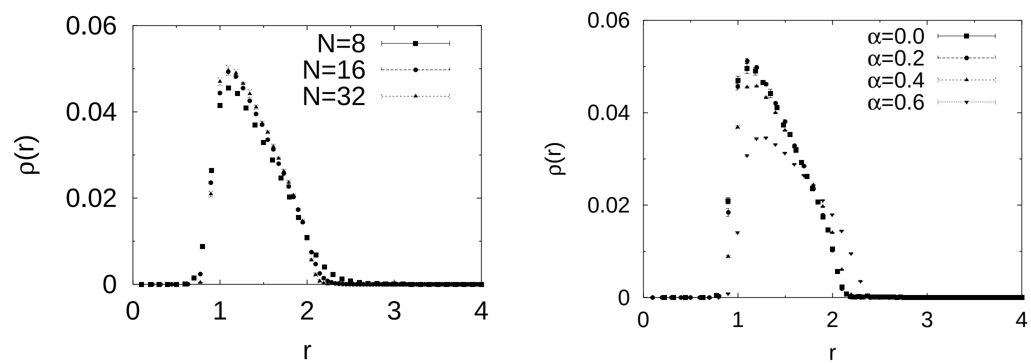

Figure 15: The plots of the eigenvalue density against the radius r, for $N = 8, 16, 32$ with $\alpha = 0.0$ to indicate the N dependence(left), and $\alpha = 0.0, 0.2, 0.4, 0.6$ and $N = 32$ to indicate the α dependence(right).

distance from the origin. $\rho(r)$ is defined as

$$\rho(r) = \frac{(\text{distribution of } \sqrt{(\text{eigenvalues})} \text{ of } Q)}{4\pi r^2}, \tag{5.16}$$

so that this may represent the eigenvalue density. $\rho(r)$ is normalized so that $\int_0^{\infty} dr 4\pi r^2 \rho(r) = 1$.

The histogram for $\alpha = 0.0$ is devoid of the eigenvalue in the vicinity of the origin, and the histogram soars around the radius 0.8. In this sense, there is an ultraviolet cutoff in the eigenvalue distribution. This phenomenon is interpreted in the following way. If a certain state is an eigenstate of the Casimir Q with almost zero eigenvalues, this implies that the same state is almost an eigenstate of A_1, A_2 and A_3 individually with almost zero eigenvalues. However, this violates the uncertainty principle, because a generic configuration of A_{μ} do not commute with each other. In the bosonic IIB matrix model [13], the vacuum expectation value $\langle \frac{1}{N} Tr F_{\mu\nu}^2 \rangle$ is of the order $\mathcal{O}(1)$. Therefore, the scale of the uncertainty $[A_{\mu}, A_{\nu}]$ is also of the order $\mathcal{O}(1)$. The above histogram is consistent with this consideration.

The histogram for $\alpha = 0.0, 0.2, 0.4, 0.6$ with $N = 32$ demonstrates that the eigenvalue distribution undergoes a gradual change as we vary α within the Yang-Mills phase.

5.2.2 Phase transition and the hysteresis

We next investigate the phase transition of the simulation initiated from (5.15). We find that the critical point differs in the initial condition (5.15) from that for the fuzzy sphere initial condition (5.2). When we initiate the simulation from $A_{\mu}^{(0)} = 0$, the critical point is found at

$$\alpha_{cr}^{(u)} \sim 0.66. \tag{5.17}$$

We plot $\langle \frac{1}{N} Tr A_{\mu}^2 \rangle$ for $N = 8, 16, 24$ with the two initial condition (5.2) and (5.15) in Fig. 16. The important difference in the Yang-Mills phase is that the critical point $\alpha_{cr}^{(u)}$ is independent of N. Since the

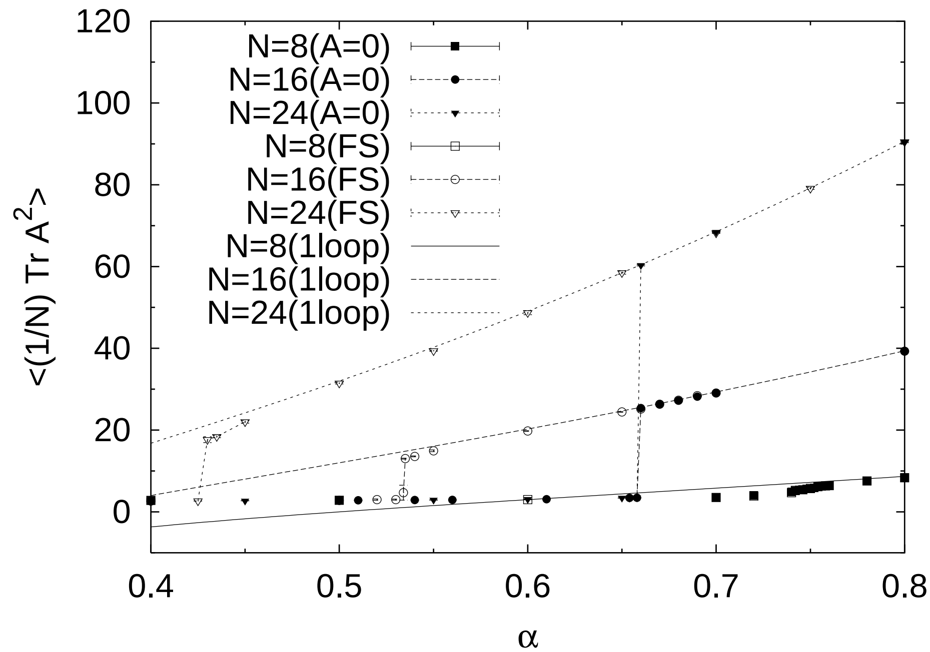

Figure 16: The plot of $\langle \frac{1}{N} Tr A_\mu^2 \rangle$ against α, for $N = 8, 16, 24$.

critical point $\alpha_{cr}^{(u)}$ is larger than that for the fuzzy sphere initial condition $\alpha_{cr}^{(l)} \sim \frac{2.1}{\sqrt{N}}$ for the sufficiently large N, we call the critical point for the Yang-Mills (fuzzy sphere) phase 'the upper (lower) critical point' respectively. The above hysteresis cycle is realized for the sufficiently large N. For the $N = 8$ case, the lower critical point is approximately $\alpha_{cr}^{(l)} \sim \frac{2.1}{\sqrt{8}} = 0.742\cdots$, which is larger than $\alpha_{cr}^{(u)}$. In this case, the hysteresis disappears, as is discerned from Fig. 16. On the other hand, this is not the case with $N = 16, 24$ in which the hysteresis cycle is realized.

5.3 Metastability of the multi-sphere solutions

In this section, we investigate the multi-fuzzy-sphere solution of the matrix model (5.1). It accommodates the following multi-fuzzy-sphere solution

$$A_\mu = \alpha \times \mathrm{diag}(L_\mu^{(n_1)}, L_\mu^{(n_2)}, \cdots, L_\mu^{(n_k)}). \tag{5.18}$$

Here, $L_\mu^{(n_a)}$ is the n_a-dimensional irreducible representation of the $SU(2)$ Lie algebra. The whole size of the matrices A_μ is $N = \sum_{a=1}^{k} n_a$. For this multi-fuzzy-sphere solution, the eigenvalue distribution of Q is peaked at

$$r_a^2 = \frac{\alpha^2}{4}(n_a^2 - 1), \quad (a = 1, 2, \cdots, k). \tag{5.19}$$

The classical value of the action is easily calculated as

$$S = -\frac{\alpha^4 N}{24} \sum_{a=1}^{k} (n_a^3 - n_a). \tag{5.20}$$

The value of the action (5.20) for the multi-fuzzy-sphere condition is bigger than that for the single fuzzy sphere state (namely, $S = -\frac{\alpha^4 N^2 (N^2-1)}{24}$). Therefore, the multi-fuzzy-sphere state can be realized as a metastable state. In the following, we report the results of the simulations to unravel this metastability.

5.3.1 Evolution of the multi-fuzzy-sphere solution

Firstly, we report the simulation with the initial condition (5.15), and observe how the fuzzy sphere state evolves. Here, we focus on the large-α case in the fuzzy sphere phase $N = 16$ and $\alpha = 2.0$. We plot the history of the vacuum expectation value of the action $\langle S \rangle$, and the eigenvalues of Q against the sweeping time in Fig. 17. This indicates the process in which the multi-fuzzy-sphere state is generated dynamically.

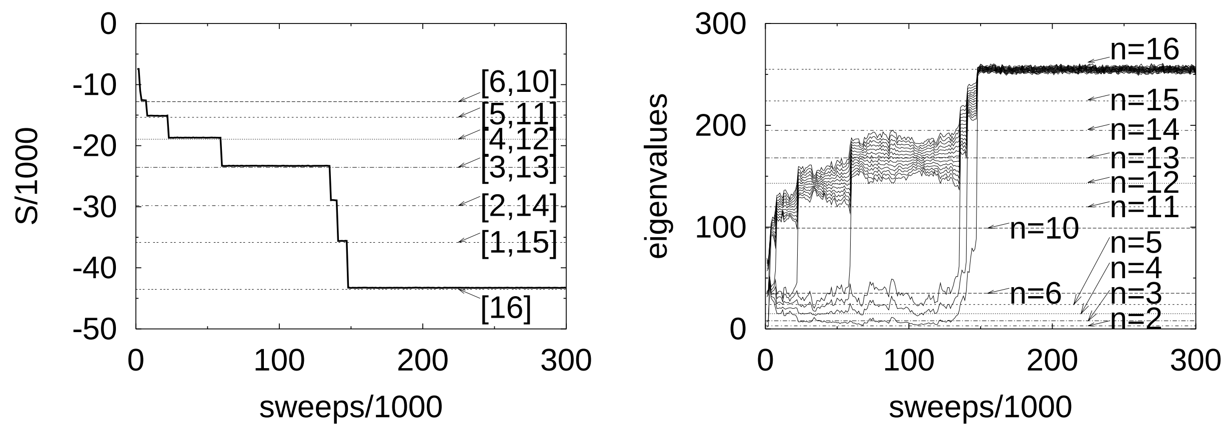

Figure 17: The history of the vacuum expectation value of the action $\langle S \rangle$ (left), and the eigenvalues of Q (right) against the sweeping time, for $N = 16$, $\alpha = 2.0$.

Firstly, we start to see a tiny plateau in the graph of $\langle S \rangle$ corresponding to the multi-fuzzy-sphere state $A_\mu = \alpha \begin{pmatrix} L_\mu^{(6)} & 0 \\ 0 & L_\mu^{(10)} \end{pmatrix}$. When we further undergo the sweeps, this multi-fuzzy-sphere state falls into the lower-energy state $A_\mu = \alpha \begin{pmatrix} L_\mu^{(5)} & 0 \\ 0 & L_\mu^{(11)} \end{pmatrix}$. In this way, the state falls off to the lower-energy state $A_\mu = \alpha \begin{pmatrix} L_\mu^{(4,3,2,1)} & 0 \\ 0 & L_\mu^{(12,13,14,15)} \end{pmatrix}$.

Finally the system arrives at the one-fuzzy-sphere irreducible representation $A_\mu = \alpha L_\mu^{(16)}$. Namely, the one-fuzzy-sphere state is dynamically generated in the course of the thermalization. When we further undergo the sweeps, the system is stuck in this one-fuzzy-sphere representation, and never falls off to the other state any more. This observation reinforces the stability of the one-fuzzy-sphere state in the fuzzy sphere phase.

5.3.2 Metastability of the fuzzy sphere solution

In this section, we obtain an insight into the metastability of the fuzzy sphere state, by focusing on the dependence on k, α and N. We adopt the initial condition of the $n = n_1 = n_2 = \cdots n_k = \frac{N}{k}$ case in (5.18) for brevity. Namely, we initiate the simulation from the multi-fuzzy-sphere state

$$A_\mu^{(0)} = \alpha L_\mu^{(n)} \otimes \mathbf{1}_{k\times k}. \tag{5.21}$$

We have observed in the previous section that the multi-fuzzy-sphere state falls into the lower-energy state in the course of the thermalization of the system. Firstly, we observe how the multi-fuzzy-sphere classical solution (5.21) decays, for $N = 16$, $\alpha = 10.0$ and $k = 2, 4, 8$ in Fig. 18. This indicates that the multi-fuzzy-sphere state (5.21) decays faster for larger k (namely, for the smaller representation of the size $n = \frac{N}{k}$). This is a natural result, and we elaborate on this dependence on k later.

Nextly, we show the first-order phase transition and the one-loop dominance of the $k = 2$ reducible representation. The observables $\frac{\langle S \rangle}{N^2}$, $\frac{1}{N}\langle \frac{1}{N} Tr A_\mu^2 \rangle$, $\langle \frac{1}{N} Tr F_{\mu\nu}^2 \rangle$ and $\frac{1}{\sqrt{N}}\langle M \rangle$ are calculated in the appendices of [60] as

$$\frac{\langle S \rangle}{N^2} = -\frac{\tilde{\alpha}^4}{24k^2} + 1, \quad \frac{1}{N}\langle \frac{1}{N} Tr A_\mu^2 \rangle = \frac{\tilde{\alpha}^2}{4k^2} - \frac{1}{\tilde{\alpha}^2}, \quad \langle \frac{1}{N} Tr (F_{\mu\nu})^2 \rangle = \frac{\tilde{\alpha}^2}{2k^2}, \quad \frac{1}{\sqrt{N}}\langle M \rangle = -\frac{\tilde{\alpha}^3}{6k^2} + \frac{1}{\tilde{\alpha}}. \tag{5.22}$$

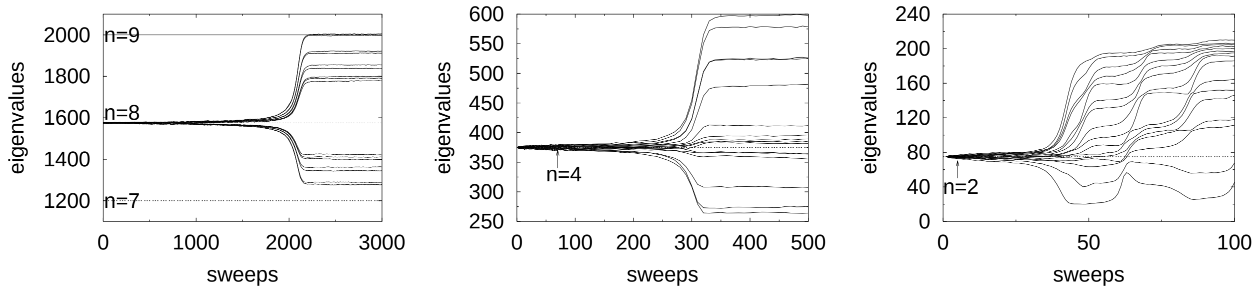

Figure 18: The eigenvalue distribution of the Casimir Q, as the multi-fuzzy-sphere solution (5.21) starts to decay, for $N = 16$, $\alpha = 10.0$ and $k = 2$ (left), $k = 4$ (middle), $k = 8$ (right).

for the one-loop effect around the multi fuzzy sphere (5.18) with $n = n_1 = n_2 = \cdots = n_k = \frac{N}{k}$. We launch the simulation from the initial condition (5.21) for $k = 2$. We plot these four observables in Fig. 19.

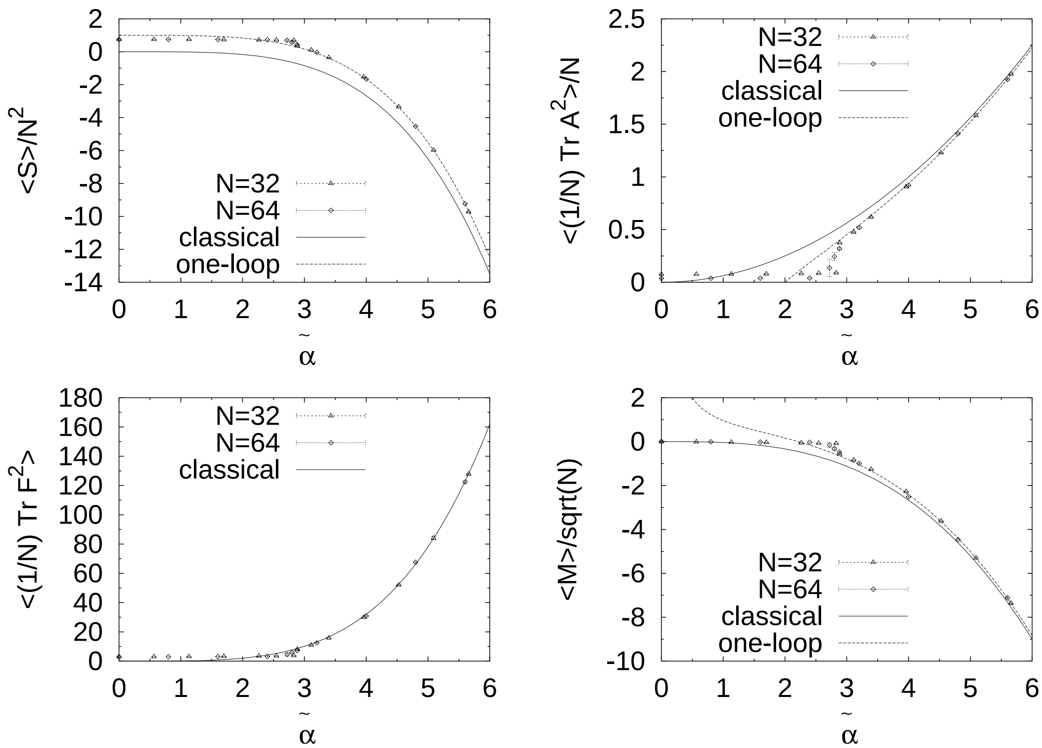

Figure 19: $\frac{1}{N^2}\langle S\rangle$ (upper left), $\frac{1}{N}\langle\frac{1}{N}TrA_\mu^2\rangle$ (upper right), $\langle\frac{1}{N}TrF_{\mu\nu}^2\rangle$ (lower left) and $\frac{1}{\sqrt{N}}\langle M\rangle$ (lower right) against $\tilde{\alpha}$, for $N = 32, 64$ for the multi fuzzy sphere state (5.21).

We have seen in Fig. 18 that the multi fuzzy sphere decays in the course of the thermalization. Here we undergo a sufficiently short sweep in the fuzzy sphere phase, such that the multi fuzzy sphere (5.21) may not decay. In the $k = 2$ multi fuzzy sphere, we find two consequenses similar to the $k = 1$ irreducible representation. Firstly, we note that there is a first-order phase transition. This time, the critical point lies at

$$\tilde{\alpha}_{cr}^{(l)k=2} \sim 2.8. \tag{5.23}$$

Secondly, the $k = 2$ multi fuzzy sphere has the one-loop dominance in the fuzzy sphere phase in the large-N limit.

We can understand the critical point (5.23) from the one-loop dominance. In the fuzzy sphere phase, the spacetime extent $\langle \frac{1}{N} Tr A_\mu^2 \rangle$ behaves at the one-loop level as $\frac{1}{N}\langle \frac{1}{N} Tr A_\mu^2 \rangle = \frac{\tilde{\alpha}^2}{4k^2} - \frac{1}{\tilde{\alpha}^2}$ in the fuzzy sphere phase. This is positive only when $\tilde{\alpha} > \sqrt{2k}$. In this sense, we obtain the lower bound on the critical point from the one-loop dominance as $\tilde{\alpha}_{cr}^{(l)k} \sim \mathcal{O}(\sqrt{k})$. The critical point (5.11) for $k = 1$ and (5.23) for $k = 2$ are consistent with this observation.

We next discuss the sweeping time τ that takes for the initial condition (5.21) to decay more qualitatively. To this end, we plot $\log(\tau k^3)$ against $\log \alpha$ for $N = 8, 16$ in Fig. 20. This indicates that the sweeping time for the multi-fuzzy-sphere state to decay depends on α and k as

$$\tau \sim \alpha^{\frac{4}{3}} k^{-3}. \tag{5.24}$$

Fig. 20 indicates that this power law is independent of the size of the matrix N.

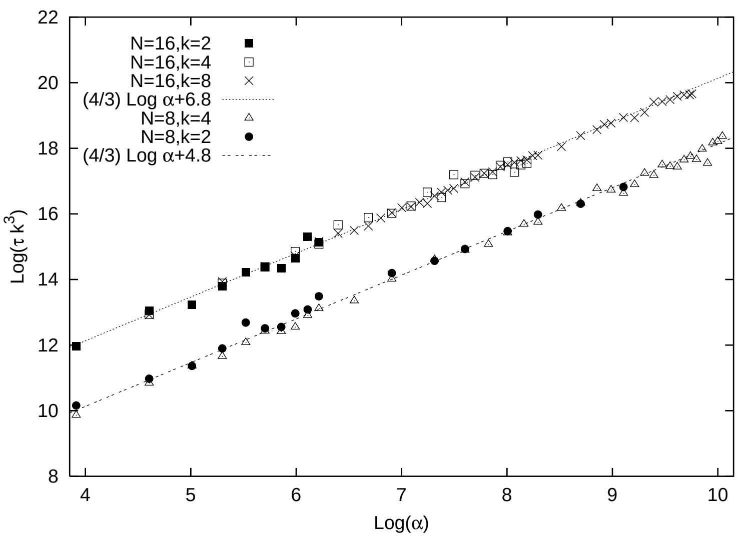

Figure 20: The plot of $\log(\tau k^3)$ against $\log \alpha$ for $N = 8, 16$.

This is in contrast to the intuitive observation that the decay probability P obeys $P \sim e^{-S} = \exp(-N^4\alpha^4 k^{-2})$; namely the sweeping time for the decay is $\frac{1}{P} \sim \exp\left(N^4\alpha^4 k^{-2}\right)$. It is interesting to ruminate on the reasoning for the deviation from this intuitive observation.

5.4 Miscellaneous future directions

We conclude this section by listing the future approaches to this work. There are a lot of interesting future works in this direction. Firstly, this toy model is expected to serve for the dynamical generation of the gauge group. As we have reviewed in Section 2.4.1, the expansion around the reducible representation (5.18) for $n_1 = \cdots n_k = \frac{N}{k}$ gives the noncommutative Yang-Mills theory with the $U(k)$ gauge group. Iso and Kawai [14] suggested that the dynamical generation of the gauge group is ascribed to the cluster distribution of the eigenvalues. By examining the eigenvalue distribution of this model, we may gain insight into the dynamical generation of the gauge group.

Secondly, it is also exciting to extend our analysis to the supersymmetric case. For the simplest case, we start with the four-dimensional supersymmetric model with the three-dimensional Chern-Simons term[16]. It is intriguing to investigate how the supersymmetry affects the results we have obtained for the bosonic case.

Thirdly, the extension to the higher-dimensional extension is also an intriguing issue. We can extend our analysis to the higher-dimensional fuzzy sphere, as we have reviewed in Section 2.4.2. There has been hitherto no works which discuss the quantum stability of the higher-dimensional fuzzy sphere S^{2k}, since it is much more mathematically involved than the simplest S^2 fuzzy sphere. On the other hand, it

[16]Austing and Wheater [57] corroborated that the three-dimensional action (2.76) has a divergent path-integral. Therefore, the numerical simulation of the three-dimensional model is not possible.

is easy to extend the algorithm for the matrix model with the three-dimensional Chern-Simons term to the higher dimension. We expect that the Monte Carlo simulation may open the door to the study of the higher-dimensional fuzzy sphere.

The fourth future direction is to extend our analysis to the general homogeneous space, which is proposed by Kitazawa in [44]. By investigating several homogeneous spaces (such as $CP^2 = SU(3)/U(2)$), we may be able to understand which homogeneous space is the most favored in the quantum sense. We expect that this direction may give us a clue to the compactification of the spacetime.

In this way, the Monte Carlo simulation we have described here gives us a lot of exciting prospects.

6 Conclusion

In this thesis, we have reviewed the author's works [26, 38, 46, 60] about the relation between the gravitational interaction and the matrix model. Now, it is believed that the 'Theory of Everything', which unifies all the interactions in the nature, is realized by the superstring theory. The large-N reduced model is now regarded as the promising candidate for the nonperturbative formulation of the superstring theory. If we build the nonperturbative formulation of the superstring theory, this will be the right framework to unify all interactions. Then, we will be able to solve all the riddles of the Standard Model, such as the dimensionality of our spacetime, the generation of the quark, the gauge group and the eighteen parameters of the Standard Model, from the parameterless theory.

In order to arrive at this ultimate goal, it is an important step to see the correspondence between the matrix model and the gravitational interaction. In the future, it is indispensable to elucidate the more manifest correspondence with the gravitational interaction. For example, it is interesting to pursue the matrix model that is equipped with the local Lorentz symmetry and reduces to the supergravity in the low-energy limit, by inheriting the idea we described in Section 4. If we find such a matrix model, it will be certainly a better extension of the IIB matrix model, in the sense that it has clearer relation to the gravitational interaction. In addition, it is also interesting how the matrix model realizes the curved-space background. The curved spacetime is also an essential feature of the general relativity. So far, it is known that the IIB matrix model, per se, realizes the curved-space background through the condensation of the graviton. In addition, several alterations of the IIB matrix model have been advocated, so that they may be equipped with the curved-space manifold. On the other hand, the alterations of the IIB matrix model is merely limited to the simple manifolds; such as the S^{2k} fuzzy sphere, fuzzy torus or CP^2 manifold. It is exciting to pursue how the matrix model realizes the general curved spacetime more manifestly.

We finally mention the relation between the IIB matrix model and its miscellaneous extensions. In the quantum field theory, some different models which share the same symmetry are equivalent in the continuum limit, known as universality. We expect that the similar mechanism may hold true of the matrix models and hence that various matrix models may have the same large N limit. Thus, there is a possibility that these new extensions are equivalent to the IIB matrix model. We can expect that both the IIB matrix model and the new extensions are equally authentic constructive definition of superstring theory in the large N limit. It is an intriguing future work to elaborate on this conjecture of the universality.

Mankind has yet to grasp what is the true 'Theory of Everything'. We must exert more effort to arrive at the answer to this ultimate and most difficult question of the elementary particle physics.

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A Notation

A.1 Definition and properties of the gamma matrices and the fermions

Basically, we follow the same notation as for [27]. Throughout this thesis, the metric of the Minkowskian spacetime is taken so that the time component should be minus and the space components should be plus. Namely, the metric is defined as

$$\eta^{\mu\nu} = \mathrm{diag}(-1, +1, \cdots, +1). \tag{A.1}$$

The gamma matrices are defined so that it may comply with the following Clifford algebra:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}_{32\times 32}. \tag{A.2}$$

Here, the gamma matrices Γ^μ are the 32×32 real matrices and the indices $\mu, \nu, \cdots$ run over $0, 1, \cdots, 9$. The explicit components are given by

$$\Gamma^0 = \mathbf{1}_{16\times 16} \otimes (-i\sigma_2) = \begin{pmatrix} 0 & -\mathbf{1}_{16\times 16} \\ \mathbf{1}_{16\times 16} & 0 \end{pmatrix}, \ \ \Gamma^p = \gamma^p \otimes \sigma_3 = \begin{pmatrix} \gamma^p & 0 \\ 0 & \gamma^p \end{pmatrix}.$$

Here, γ^p are the nine-dimensional (and thus 16×16) gamma matrices, and obey the Clifford algebra for the nine-dimensional Euclidean spacetime $\{\gamma^p, \gamma^q\} = 2\delta^{pq}\mathbf{1}_{16\times 16}$, where $p, q, \cdots$ run over $1, 2, \cdots, 9$.

The transpose of the gamma matrices are given by

$${}^T(\Gamma^0) = -\Gamma^0, \ \ {}^T(\Gamma^p) = +\Gamma^p \ \ (p = 1, 2, \cdots, 9).. \tag{A.3}$$

The chirality matrices $\Gamma^{10} (= \Gamma^\sharp)$ are defined by

$$\Gamma^\sharp = \Gamma^0\Gamma^1\Gamma^2\cdots\Gamma^9 = \mathbf{1}_{16\times 16} \otimes \sigma_1 = \begin{pmatrix} 0 & \mathbf{1}_{16\times 16} \\ \mathbf{1}_{16\times 16} & 0 \end{pmatrix}. \tag{A.4}$$

For the indices $A, B, \cdots = 0, 1, \cdots, 10$ in the eleven-dimensional Minkowski spacetime, the gamma matrices comply with the Clifford algebra $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}\mathbf{1}_{32\times 32}$.

Next, we define the charge conjugation matrix C. C should satisfy the conditions

$$C\Gamma^\mu = -{}^T(\Gamma^\mu)C, \ \ C +^T C = 0. \tag{A.5}$$

In our case, C is identical to Γ^0.

The anti-symmetrized gamma matrices are abbreviated as

$$\Gamma^{A_1\cdots A_k} = \frac{1}{k!}\sum_{\sigma\in\mathcal{S}_n} \mathrm{sgn}(\sigma)\Gamma^{A_{\sigma(1)}}\Gamma^{A_{\sigma(2)}}\cdots\Gamma^{A_{\sigma(k)}}. \tag{A.6}$$

Here, $\mathcal{S}_n$ is the set of the permutation of the integers $1, 2, \cdots, k$.

A.1.1 Chirality of the Weyl fermion

We summarize the notation for the chirality of the fermion. The left-hand (right-hand) fermion is defined as

$$\psi_L = \frac{1+\Gamma^\sharp}{2}\psi, \ \ \psi_R = \frac{1-\Gamma^\sharp}{2}\psi. \tag{A.7}$$

The left-handed (right-handed) fermions satisfy the following identities for the ten-dimensional gamma matrices:

$$\bar{\chi}_L \Gamma^{\mu_1 \cdots \mu_{2k}} \epsilon_L = \bar{\chi}_R \Gamma^{\mu_1 \cdots \mu_{2k}} \epsilon_R = 0, \;\; \bar{\chi}_L \Gamma^{\mu_1 \cdots \mu_{2k+1}} \epsilon_R = \bar{\chi}_R \Gamma^{\mu_1 \cdots \mu_{2k+1}} \epsilon_L = 0. \tag{A.8}$$

Here, we prove only $\bar{\chi}_L \Gamma^{\mu_1 \cdots \mu_{2k}} \epsilon_L = 0$, so that the rest can be verified likewise.

$$\begin{aligned} \bar{\chi}_L \Gamma^{\mu_1 \cdots \mu_{2k}} \epsilon_L &= {}^T\chi^T (\frac{1+\Gamma^\sharp}{2}) \Gamma^0 \Gamma^{\mu_1 \cdots \mu_{2k}} \frac{1+\Gamma^\sharp}{2} \epsilon = {}^T\chi \frac{1+\Gamma^\sharp}{2} \frac{1+(-1)^{2k+1}\Gamma^\sharp}{2} \Gamma^0 \Gamma^{\mu_1 \cdots \mu_{2k}} \epsilon \\ &= {}^T\chi \frac{1+\Gamma^\sharp}{2} \frac{1-\Gamma^\sharp}{2} \Gamma^0 \Gamma^{\mu_1 \cdots \mu_{2k}} \epsilon = 0. \end{aligned} \tag{A.9}$$

A.1.2 Epsilon tensors and the duality of the matrices

Here, we define the anti-symmetric epsilon tensors as

$$\epsilon_{01 \cdots k} = 1, \;\; (\text{so that } \epsilon^{01 \cdots k} = -1). \tag{A.10}$$

This definition leads the duality relations of the gamma matrices

$$\Gamma^{A_0 \cdots A_k} = \frac{(-1)^{\frac{k(k+1)}{2}}}{(10-k)!} \epsilon^{A_0 \cdots A_9 A_\sharp} \Gamma_{A_{k+1} \cdots A_\sharp}. \tag{A.11}$$

A.1.3 Multiplication law of the gamma matrices

The anti-symmetrized gamma matrices obey the following formula of the product.

$$\begin{aligned} \Gamma^{A_1 \cdots A_m} \Gamma^{B_1 \cdots B_n} &= \Gamma^{A_1 \cdots A_m B_1 \cdots B_n} + (-1)^{m-1} {}_m C_1 {}_n C_1 \eta^{[A_1}{}^{[B_1} \Gamma^{A_2 \cdots A_m]}{}^{B_2 \cdots B_n]} \\ &+ (-1)^{(m-1)+(m-2)} {}_m C_2 {}_n C_2 2! \eta^{[A_1}{}^{[B_1} \eta^{A_2}{}^{B_2} \Gamma^{A_3 \cdots A_m]}{}^{B_3 \cdots B_n]} \\ &+ (-1)^{(m-1)+(m-2)+(m-3)} {}_m C_3 {}_n C_3 3! \eta^{[A_1}{}^{[B_1} \eta^{A_2}{}^{B_2} \eta^{A_3}{}^{B_3} \Gamma^{A_4 \cdots A_m]}{}^{B_4 \cdots B_n]} + \cdots. \end{aligned} \tag{A.12}$$

A.1.4 Flipping property of the Majorana fermions

Here, we introduce a useful formula for the flipping of the Majorana fermion. The gamma matrices has the following property:

$$\Gamma^0 ({}^T\Gamma^A) \Gamma^0 = \Gamma^A. \tag{A.13}$$

This leads to the following relation:

$$\begin{aligned} \Gamma^0 ({}^T\Gamma^{A_1 \cdots A_k}) \Gamma^0 &= (-1)^{k-1} (\Gamma^0 ({}^T\Gamma^{A_k}) \Gamma^0) \cdots (\Gamma^0 ({}^T\Gamma^{A_1}) \Gamma^0) = (-1)^{k-1} \Gamma^{A_k \cdots A_1} \\ &= (-1)^{k-1} (-1)^{\frac{k(k-1)}{2}} \Gamma^{A_1 \cdots A_k} = (-1)^{\frac{(k+2)(k-1)}{2}} \Gamma^{A_1 \cdots A_k}. \end{aligned} \tag{A.14}$$

Then, the Majorana fermions are flipped as

$$\begin{aligned} \bar{\chi} \Gamma^{A_1 \cdots A_k} \epsilon &= {}^T(\bar{\chi} \Gamma^{A_1 \cdots A_k} \epsilon) = -{}^T\epsilon ({}^T\Gamma^{A_1 \cdots A_k}) ({}^T\Gamma^0) \chi = -{}^T\epsilon (\Gamma^0)^2 ({}^T\Gamma^{A_1 \cdots A_k}) (\Gamma^0) \chi \\ &= (-1)^{\frac{k(k+1)}{2}} \bar{\epsilon} \Gamma^{A_1 \cdots A_k} \chi. \end{aligned} \tag{A.15}$$

Namely, the sign is minus for $k = 1, 2, 5$, and plus for $k = 0, 3, 4$.

A.1.5 Proof of the Fierz identity (2.40) and (2.87)

We give a proof of the formula of the Fierz identity (2.40) and (2.87), which contributes to the proof of the supersymmetry algebra of the matrix model.

We start with the ten-dimensional formula (2.40). The fermions ϵ_1 and ϵ_2 are Majorana-Weyl fermions, and effectively act on the 16×16 space of the gamma matrices. The left hand side is written as

$$\bar{\epsilon}_1 \Gamma_\nu \psi \Gamma^{\mu\nu} \epsilon_2 = (\bar{\epsilon}_1)^\alpha (\Gamma_\nu)_\alpha{}^\beta \psi_\beta (\Gamma^{\mu\nu})_\gamma{}^\delta \epsilon_{2\delta} = -(\Gamma^{\mu\nu})_\gamma{}^\delta (\bar{\epsilon}_1 \epsilon_2)_\delta{}^\alpha (\Gamma_\nu)_\alpha{}^\beta \psi_\beta. \tag{A.16}$$

Here, the minus sign emerges because we have flipped the order of the Grassmann odd fermions. The matrix $(\bar{\epsilon}_1\epsilon_2)_\delta{}^\alpha$ can be decomposed as

$$
\begin{aligned}
(\bar{\epsilon}_1\epsilon_2)_\delta{}^\alpha &= \frac{1}{16}(\bar{\epsilon}_1\epsilon_2)\mathbf{1} + \frac{1}{16}(\bar{\epsilon}_1\Gamma_\mu\epsilon_2)\Gamma^\mu - \frac{1}{16\times 2!}(\bar{\epsilon}_1\Gamma_{\mu_1\mu_2}\epsilon_2)\Gamma^{\mu_1\mu_2} - \frac{1}{16\times 3!}(\bar{\epsilon}_1\Gamma_{\mu_1\mu_2\mu_3}\epsilon_2)\Gamma^{\mu_1\mu_2\mu_3} \\
&+ \frac{1}{16\times 4!}(\bar{\epsilon}_1\Gamma_{\mu_1\cdots\mu_4}\epsilon_2)\Gamma^{\mu_1\cdots\mu_4} + \frac{1}{16\times 5!}(\bar{\epsilon}_1\Gamma_{\mu_1\cdots\mu_5}\epsilon_2)\Gamma^{\mu_1\cdots\mu_5}. \qquad \text{(A.17)}
\end{aligned}
$$

Since the fermions ϵ_1 and ϵ_2 have the same chirality, the terms of the even rank vanish due to the relation (A.7). We ignore the rank-3 term because we are interested in the difference $\bar{\epsilon}_1\Gamma_\nu\psi\Gamma^{\mu\nu}\epsilon_2 - \bar{\epsilon}_2\Gamma_\nu\psi\Gamma^{\mu\nu}\epsilon_1$, and the rank-3 term does not contribute due to the relation (A.15). This leads us to focus only on the rank-1,5 terms. The relevant gamma matrices are computed as

$$
\begin{aligned}
\Gamma^{\mu\nu}\Gamma_\rho\Gamma_\nu &= \Gamma^{\mu\nu}(\Gamma_{\rho\nu} + \eta_{\rho\nu}) = (-\eta_\nu{}^\nu\Gamma^\mu{}_\rho + 2\Gamma^\mu{}_\rho - \eta_\nu{}^\nu\eta^\mu{}_\rho + \eta^\mu{}_\rho) + \Gamma^\mu{}_\rho \overset{\eta^\nu{}_\nu=10}{=} 7\Gamma_\rho{}^\mu - 9\eta^\mu{}_\rho \\
&= 7(\Gamma_\rho\Gamma^\mu - \eta^\mu{}_\rho) - 9\eta^\mu{}_\rho = 7\Gamma_\rho\Gamma^\mu - 16\eta^\mu{}_\rho, \qquad \text{(A.18)} \\
\Gamma^{\mu\nu}\Gamma_{\rho_1\cdots\rho_5}\Gamma_\nu &= \Gamma^{\mu\nu}(-\Gamma_{\nu\rho_1\cdots\rho_5} + 5\eta_{\nu[\rho_1}\Gamma_{\rho_2\cdots\rho_5]}) \\
&= -(-\eta^{\mu\nu}\Gamma_{\nu\rho_1\cdots\rho_5} + \eta^\nu{}_\nu\Gamma^\mu{}_{\rho_1\cdots\rho_5} - 5\eta_{\nu[\rho_1}\Gamma^{\mu\nu}{}_{\rho_2\cdots\rho_5]}) + 5(\Gamma^\mu{}_{\rho_1\cdots\rho_5} - \eta^\mu{}_{[\rho_2}\Gamma_{\rho_1\rho_3\rho_4\rho_5]}) \\
&= -\Gamma_{\rho_1\cdots\rho_5}{}^\mu + 5\eta^\mu{}_{[\rho_1}\Gamma_{\rho_2\cdots\rho_5]} = \Gamma_{\rho_1\cdots\rho_5}\Gamma^\mu. \qquad \text{(A.19)}
\end{aligned}
$$

When we substitute (A.17), (A.18) and (A.19) into (A.16), the computation of the Fierz transformation goes as follows:

$$
\begin{aligned}
\bar{\epsilon}_1\Gamma_\nu\psi\Gamma^{\mu\nu}\epsilon_2 &= -(\Gamma^{\mu\nu})_\gamma{}^\delta(\bar{\epsilon}_1\epsilon_2)_\delta{}^\alpha(\Gamma_\nu)_\alpha{}^\beta\psi_\beta \\
&= -\frac{1}{16}(\bar{\epsilon}_1\Gamma_\rho\epsilon_2)(\Gamma^{\mu\nu}\Gamma^\rho\Gamma_\nu)\psi - \frac{1}{16\times 5!}(\bar{\epsilon}_1\Gamma_{\rho_1\cdots\rho_5}\epsilon_2)(\Gamma^{\mu\nu}\Gamma^{\rho_1\cdots\rho_5}\Gamma_\nu)\psi + (\text{rank 3 term}) \\
&= \bar{\epsilon}_1\Gamma^\mu\epsilon_2\psi - \frac{7}{16}\bar{\epsilon}_1\Gamma^\rho\epsilon_2\Gamma_\rho\Gamma^\mu\psi - \frac{1}{16\times 5!}\bar{\epsilon}_1\Gamma^{\rho_1\cdots\rho_5}\epsilon_2\Gamma_{\rho_1\cdots\rho_5}\Gamma^\mu\psi + (\text{rank 3 term}). \qquad \text{(A.20)}
\end{aligned}
$$

We next go to the proof of (2.87) for the three-dimensional spacetime. This time, the fermions are Majorana ones. The proof goes in the same way as for (2.40). Here, we note that the 2×2 matrices M are decomposed by the Paulian matrices as $M = \frac{1}{2}(TrM)\mathbf{1} + \frac{1}{2}(TrM\sigma_\mu)\sigma_\mu$. Then, the Fierz decomposition is verified as follows:

$$
\bar{\epsilon}_1\sigma_\mu\psi\sigma_{\mu\nu}\epsilon_2 = (\bar{\epsilon}_1)^\alpha(\sigma_\mu)_\alpha{}^\beta\psi_\beta(\sigma_{\mu\nu})_\gamma{}^\delta(\epsilon_2)_\delta = -(\sigma_{\mu\nu})_\gamma{}^\delta(\bar{\epsilon}_1\epsilon_2)_\delta{}^\alpha(\sigma_\mu)_\alpha{}^\beta\psi_\beta \sim -\frac{1}{2}(\bar{\epsilon}_1\sigma_\chi\epsilon_2)(\sigma_{\mu\nu}\sigma_\chi\sigma_\mu)\psi. \qquad \text{(A.21)}
$$

Here, we drop the contribution of the rank-0 term $(\bar{\epsilon}_1\mathbf{1}\epsilon_2)$, because we are interested in the difference $\bar{\epsilon}_1\sigma_\mu\psi\sigma_{\mu\nu}\epsilon_2 - \bar{\epsilon}_2\sigma_\mu\psi\sigma_{\mu\nu}\epsilon_1$ and the rank-0 term cancels. Likewise, the latter formula is rewritten as

$$
\bar{\epsilon}_1\sigma_\mu\psi\sigma_\mu\epsilon_2 = (\bar{\epsilon}_1)^\alpha(\sigma_\mu)_\alpha{}^\beta\psi_\beta(\sigma_\mu)_\gamma{}^\delta(\epsilon_2)_\delta = -(\sigma_\mu)_\gamma{}^\delta(\bar{\epsilon}_1\epsilon_2)_\delta{}^\alpha(\sigma_\mu)_\alpha{}^\beta\psi_\beta \sim -\frac{1}{2}(\bar{\epsilon}_1\sigma_\chi\epsilon_2)(\sigma_\mu\sigma_\chi\sigma_\mu)\psi. \qquad \text{(A.22)}
$$

The relevant Paulian matrices are calculated as

$$
\begin{aligned}
\sigma_{\mu\nu}\sigma_\chi\sigma_\mu &= i\epsilon_{\mu\nu\rho}\sigma_\rho(\delta_{\chi\mu} + i\epsilon_{\chi\mu\eta}\sigma_\eta) = i\epsilon_{\chi\nu\rho}\sigma_\rho - (\delta_{\nu\eta}\delta_{\rho\chi} - \delta_{\nu\chi}\delta_{\rho\eta})(\delta_{\rho\eta} + i\epsilon_{\rho\eta\mu}\sigma_\mu) \\
&= i\epsilon_{\chi\nu\rho}\sigma_\rho - i\epsilon_{\chi\nu\rho}\sigma_\rho - \delta_{\nu\chi} + \delta_{\nu\chi}\underbrace{\delta_{\rho\eta}\delta_{\rho\eta}}_{=3} = 2\delta_{\nu\chi}, \qquad \text{(A.23)} \\
\sigma_\mu\sigma_\chi\sigma_\mu &= \sigma_\mu(\delta_{\chi\mu} + i\epsilon_{\chi\mu\eta}\sigma_\eta) + \sigma_\chi + i\epsilon_{\chi\mu\eta}(i\epsilon_{\mu\eta\rho}\sigma_\rho + \delta_{\mu\eta}) = \sigma_\chi - (\delta_{\eta\eta}\delta_{\chi\rho} - \delta_{\eta\chi}\delta_{\eta\rho})\sigma_\rho \\
&= \sigma_\chi(1 - 3 + 1) = -\sigma_\chi. \qquad \text{(A.24)}
\end{aligned}
$$

Plugging (A.23) and (A.24) into (A.21) and (A.22), we complete the proof of the Fierz identity (2.87).

A.1.6 Proof of the Fierz identity (2.10)

We next verify the Fierz identity (2.10). This formula plays an important role in the supersymmetry of the super Yang-Mills theory or the Green-Schwarz action of the superstring. This holds only for the following cases:

1. $d = 3$, and $\psi_{1,2,3}$ are Majorana.
2. $d = 4$, and $\psi_{1,2,3}$ are Majorana or Weyl.
3. $d = 6$, and $\psi_{1,2,3}$ are Weyl.
4. $d = 10$, and $\psi_{1,2,3}$ are Majorana-Weyl.

To substantiate this relation, we carry out the following decomposition.

$$\begin{aligned} & (\bar{\epsilon}\Gamma^\mu\psi_1)(\bar{\psi}_2\Gamma_\mu\psi_3) + (\bar{\epsilon}\Gamma^\mu\psi_2)(\bar{\psi}_3\Gamma_\mu\psi_1) + (\bar{\epsilon}\Gamma^\mu\psi_3)(\bar{\psi}_1\Gamma_\mu\psi_2) \\ = \; & (\epsilon)_\alpha(\Gamma^0\Gamma^\mu)_{\alpha\beta}(\psi_1)_\beta(\psi_2)_\gamma(\Gamma^0\Gamma_\mu)_{\gamma\delta}(\psi_3)_\delta + (\epsilon)_\alpha(\Gamma^0\Gamma^\mu)_{\alpha\gamma}(\psi_2)_\gamma(\psi_3)_\delta(\Gamma^0\Gamma_\mu)_{\delta\beta}(\psi_1)_\beta \\ + \; & (\epsilon)_\alpha(\Gamma^0\Gamma^\mu)_{\alpha\delta}(\psi_3)_\delta(\psi_1)_\beta(\Gamma^0\Gamma_\mu)_{\beta\gamma}(\psi_2)_\gamma. \end{aligned} \tag{A.25}$$

Namely, our job reduces to verifying the vanishing of the following quantity:

$$(\Gamma^0\Gamma^\mu)_{\alpha\beta}(\Gamma^0\Gamma_\mu)_{\gamma\delta} + (\Gamma^0\Gamma^\mu)_{\alpha\gamma}(\Gamma^0\Gamma_\mu)_{\delta\beta} + (\Gamma^0\Gamma^\mu)_{\alpha\delta}(\Gamma^0\Gamma_\mu)_{\beta\gamma}. \tag{A.26}$$

We multiply $(\chi_1)_\gamma$ and $(\chi_2)_\delta$ with (A.26) to obtain[17]

$$(\Gamma^\mu)_{\alpha\beta}({}^T\chi_1\Gamma_0\Gamma_\mu\chi_2) + (\Gamma^\mu\chi_1)_\alpha({}^T\chi_2\Gamma^0\Gamma_\mu)_\beta - (\Gamma^\mu\chi_2)_\alpha({}^T\chi_1\Gamma^0\Gamma_\mu)_\beta. \tag{A.27}$$

We verify the vanishing of (A.27) by decomposing them by the gamma matrices. For example, the 16×16 matrices for the ten-dimensional case are decomposed in the same way as (A.17). This leads us to verify the vanishing for the components of each rank.

For $(\Gamma^\mu)_{\alpha\beta}$ of the first term, it is trivial that only the rank-1 components contribute. For the second and third terms which entail the fermions $\chi_{1,2}$, we can limit the argument at most to the rank-5 components. We note that the rank-0,3,4 components do not contribute because of the anti-symmetry of $\Gamma_0\Gamma_{\rho_1\cdots\rho_k}$ (for $k = 0, 3, 4$). The even-rank components do not contribute for the Weyl fermion either, because of the Weyl projection. We list up which rank we should consider in the following:

1. For $d = 3$ Majorana case, only the rank-1 matrices may contribute.
2. For $d = 4$ Majorana case, only the rank-1,2 components may survive. For the Weyl case, only the rank-1 may survive.
3. For the $d = 6$ Weyl case, we consider the rank-1 components.
4. For the $d = 10$ Majorana-Weyl case, we consider the rank-1,5 components.

For the rank-1 case, we multiply $(\Gamma_\rho)_{\beta\alpha}$, and we find that this quantity vanishes:

$$Tr(\Gamma^\mu\Gamma_\rho)({}^T\chi_1\Gamma_0\Gamma_\mu\chi_2) - ({}^T\chi_2\Gamma_0\Gamma_\mu\Gamma_\rho\Gamma^\mu\chi_1) + ({}^T\chi_1\Gamma_0\Gamma_\mu\Gamma_\rho\Gamma^\mu\chi_2) = (-\mathcal{P} + 2(d-2))({}^T\chi_1\Gamma_0\Gamma_\rho\chi_2) = 0.$$

$\mathcal{P}$ is the size of the gamma matrices, and d is the spacetime dimensionality. Here, we utilize the following formula for the general rank-k gamma matrices $\Gamma_{\rho_1\cdots\rho_k}$:

$$\begin{aligned} \Gamma_\mu\Gamma_{\rho_1\cdots\rho_k}\Gamma^\mu &= (\Gamma_{\mu\rho_1\cdots\rho_k} + k\eta_{\mu[\rho_1}\Gamma_{\rho_2\cdots\rho_k]})\Gamma^\mu = (-1)^{k-1}(-\eta^\mu_\mu\Gamma_{\rho_1\cdots\rho_k} + k\eta^\mu_{[\rho_1}\Gamma_{\mu\rho_2\cdots\rho_k]}) + k\eta_{\mu[\rho_1}\Gamma_{\rho_2\cdots\rho_k]\mu} \\ &= (-1)^k(d-2k)\Gamma_{\rho_1\cdots\rho_k}. \end{aligned} \tag{A.28}$$

The rank-1 terms cancel for $d = 3, 4, 6, 10$, where $\mathcal{P} = 2(d-2) = 2, 4, 8, 16$ respectively.

Next, we go on to the generic rank-k ($k = 2, 3, \cdots$) case. We multiply $(\Gamma_{\rho_1\cdots\rho_k})_{\beta\alpha}$ to obtain

$$-({}^T\chi_2\Gamma_0\Gamma_\mu\Gamma_{\rho_1\cdots\rho_k}\Gamma^\mu\chi_1) + ({}^T\chi_1\Gamma_0\Gamma_\mu\Gamma_{\rho_1\cdots\rho_k}\Gamma^\mu\chi_2) = 2({}^T\chi_1\Gamma_0\Gamma_\mu\Gamma_{\rho_1\cdots\rho_k}\Gamma^\mu\chi_2). \tag{A.29}$$

Due to the formula of the gamma matrices (A.28), these contributions actually vanish for $k = 2, 5$ for $d = 4, 10$ respectively.

[17] The flipping property $\bar{\chi}_1\Gamma_{\rho_1\cdots\rho_k}\chi_2 = (-1)^{\frac{k(k+1)}{2}}\bar{\chi}_2\Gamma_{\rho_1\cdots\rho_k}\chi_1$ holds true *only for the Majorana fermion*, because $\bar{\chi} = {}^T\chi\Gamma^0$ holds only for the Majorana fermions. However, the property ${}^T\chi_1\Gamma^0\Gamma_{\rho_1\cdots\rho_k}\chi_2 = (-1)^{\frac{k(k+1)}{2}}\,{}^T\chi_2\Gamma^0\Gamma_{\rho_1\cdots\rho_k}\chi_1$ holds true without assuming the Majorana fermion.

A.2 Supermatrices

This section is devoted to introducing the definitions of the notion of supermatrices. In treating supermatrices, there are many points we should be meticulous about, because what holds true of ordinary matrices is not applicable to the supermatrices.

A.2.1 Transpose

We first introduce a notion of the transpose, emphasizing on the difference from the ordinary matrices. In considering such objects, it is extremely important to settle the starting point, because the other notions are defined so that they are consistent with this starting point.

Transpose of Vector

The guiding principle in considering the transpose of the supermatrices is *the transpose of the vector*.

- The guiding principle is that *the transpose of a vector* is defined as

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^T = (x_1, \cdots, x_n). \tag{A.30}$$

  We denote $\{x_\mu\}$ as the components of v, and these components mean both bosons and fermions.

- We define the vector as $v = \begin{pmatrix} \eta \\ b \end{pmatrix}$ where η and b are fermionic and bosonic *real* fields respectively.

Transpose of Supermatrices

The transpose of supermatrices must be defined so that the definition is consistent with the transpose of a vector. Therefore the transpose of a supermatrix must satisfy

$${}^T(Mv) = {}^T v {}^T M, \tag{A.31}$$

where M is a supermatrix and v is a vector. Following this rule, the transpose of a supermatrix is defined as follows:

$$\text{For } M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \ \ \textit{and} \ \ v = \begin{pmatrix} \eta \\ b \end{pmatrix}, \quad {}^T M = \begin{pmatrix} {}^T a & -{}^T\gamma \\ {}^T\beta & {}^T d \end{pmatrix}. \tag{A.32}$$

- a and d are bosonic (i.e. Grassmann even) $m \times m$ and $n \times n$ matrices, respectively.
- β and γ are $m \times n$ and $n \times m$ fermionic (i.e. Grassmann odd) matrices, respectively.
- η (b) denote the upper m (lower n) bosonic(fermionic) components of the supervector respectively.

(Proof) This can be verified using the very definition of the transpose.

$$Mv = \begin{pmatrix} a\eta + \beta b \\ \gamma\eta + db \end{pmatrix}. \tag{A.33}$$

Then the transpose of this vector is by definition

$${}^T(Mv) = ({}^T\eta {}^T a + {}^T b {}^T\beta, -{}^T\eta {}^T\gamma + {}^T b {}^T d) = {}^T v {}^T M. \tag{A.34}$$

The point is that the sign of $\gamma\eta$ has changed because these are Grassmann odd. Noting this fact, we can read off the result (A.32). (Q.E.D.)

We have one caution about the transpose of the supermatrix. The transpose of the transpose does not give an original matrix. This 'anomalous' property can be immediately read off from the definition of the supermatrix (A.32):

$${}^T({}^T \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}) = {}^T \begin{pmatrix} {}^T a & -{}^T\gamma \\ {}^T\beta & {}^T d \end{pmatrix} = \begin{pmatrix} a & -\beta \\ -\gamma & d \end{pmatrix}. \tag{A.35}$$

Transpose of Transverse Vector

We have seen an important fact that the transpose of the transpose of a supermatrix does not give the original supermatrix. In fact, the same holds true of the transpose of the transpose of a vector. Conclusion coming first, the definition is

$$ {}^{T}y = {}^{T}(\eta, b) = \begin{pmatrix} -{}^{T}\eta \\ {}^{T}b \end{pmatrix}. \tag{A.36}$$

We confirm that this is actually a well-defined settlement. This notion is defined so that

$$ {}^{T}(yM) = {}^{T}M{}^{T}y, \tag{A.37}$$

where y is a transverse vector $y = (\eta, b)$ and M is a supermatrix $M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$. We compute both the L.H.S and the R.H.S and verify that they actually match if we follow the above definition.

- L.H.S. : ${}^{T}(yM) = {}^{T}(\eta a + b\gamma, \eta\beta + bd) = \begin{pmatrix} -{}^{T}(\eta a) - {}^{T}(b\gamma) \\ {}^{T}(\eta\beta) + {}^{T}(bd) \end{pmatrix} = \begin{pmatrix} -{}^{T}a{}^{T}\eta - {}^{T}\gamma{}^{T}b \\ -{}^{T}\beta{}^{T}\eta + {}^{T}d{}^{T}b \end{pmatrix}$.
  We have used the fact that in the transpose of fF, we must multiply -1 because a fermion jumps over another fermion.
- R.H.S. : ${}^{T}M{}^{T}y = \begin{pmatrix} {}^{T}a & -{}^{T}\gamma \\ {}^{T}\beta & {}^{T}d \end{pmatrix} \begin{pmatrix} -{}^{T}\eta \\ {}^{T}b \end{pmatrix} = \begin{pmatrix} -{}^{T}a{}^{T}\eta - {}^{T}\gamma{}^{T}b \\ -{}^{T}\beta{}^{T}\eta + {}^{T}\gamma{}^{T}b \end{pmatrix}$.

Thus we have verified that the above definition of the transpose is consistent with the condition ${}^{T}(yM) = {}^{T}M{}^{T}y$. As we have mentioned before, the transpose of the transpose of a vector does not give the original vector, because

$$ {}^{T}({}^{T}\begin{pmatrix} \eta \\ b \end{pmatrix}) = {}^{T}({}^{T}\eta, {}^{T}b) = \begin{pmatrix} -\eta \\ b \end{pmatrix}. \tag{A.38}$$

A.2.2 Hermitian Conjugate

We introduce a notion of hermitian conjugate of the supermatrix. This notion is much simpler than the transpose, and we do not see anomalous properties as emerged in the transpose. The starting point of this notion is

- For a fermionic *number* α, β, the complex conjugate is $(\alpha\beta)^\dagger = (\beta)^\dagger(\alpha)^\dagger$.
- For a vector $v = \begin{pmatrix} \eta \\ b \end{pmatrix}$, the complex conjugate is $\begin{pmatrix} \eta \\ b \end{pmatrix}^\dagger = (\eta^\dagger, b^\dagger)$.

Under this definition, the hermitian conjugate of a supermatrix is defined as

$$ \text{For } M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}, \quad M^\dagger = \begin{pmatrix} a^\dagger & \gamma^\dagger \\ \beta^\dagger & d^\dagger \end{pmatrix}. \tag{A.39}$$

(Proof) The guiding principle to determine the hermitian conjugate of a supermatrix is the condition

$$ (Mv)^\dagger = (v^\dagger)(M^\dagger). \tag{A.40}$$

For $M = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$ and $v = \begin{pmatrix} \eta \\ b \end{pmatrix}$, $(Mv)^\dagger$ is computed to be, utilizing the definition for the vector,

$$ (Mv)^\dagger = \begin{pmatrix} a\eta + \beta b \\ \gamma\eta + db \end{pmatrix}^\dagger = ((a\eta)^\dagger + (\beta b)^\dagger, (\gamma\eta)^\dagger + (db)^\dagger) = (\eta^\dagger a^\dagger + b^\dagger\beta^\dagger, \eta^\dagger\gamma^\dagger + b^\dagger d^\dagger). \tag{A.41}$$

The explicit form of the hermitian conjugate of a supermatrix can be read off from (A.41), and this completes the proof. (Q.E.D.)

The definition of the hermitian conjugate of a transverse vector is now straightforward. This is given by

$$ y^\dagger = (\eta, b)^\dagger = \begin{pmatrix} \eta^\dagger \\ b^\dagger \end{pmatrix}. \tag{A.42}$$

It is straightforward to verify that this definition is consistent with the guiding principle

$$(yM)^\dagger = M^\dagger y^\dagger, \tag{A.43}$$

because $(l.h.s.) = (r.h.s.) = \begin{pmatrix} a^\dagger\eta^\dagger + \gamma^\dagger b^\dagger \\ \beta^\dagger\eta^\dagger + d^\dagger b^\dagger \end{pmatrix}$.

A.2.3 Complex Conjugate

We define a notion of complex conjugate for a supermatrix. The guiding principle to define the complex conjugate is to require the matrices and the vectors to satisfy the condition

$$(Mv)^* = M^* v^*. \tag{A.44}$$

In order to satisfy this condition, we define the complex conjugate of the vectors and the matrices as follows[18].

$$v^* \overset{def}{=} ({}^T v)^\dagger, \quad M^* \overset{def}{=} ({}^T M)^\dagger. \tag{A.45}$$

It is straightforward to verify that this definition is consistent with the guiding principle (A.44).

$$(Mv)^* = (({}^T v)({}^T M))^\dagger = ({}^T M)^\dagger ({}^T v)^\dagger = M^* v^*. \tag{A.46}$$

Combining the results obtained in the previous section, the explicit form of the complex conjugate of the vectors and the matrices are

$$M^* = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}^* = \begin{pmatrix} a^* & \beta^* \\ -\gamma^* & d^* \end{pmatrix}, \; v^* = \begin{pmatrix} \eta \\ b \end{pmatrix}^* = \begin{pmatrix} \eta^* \\ b^* \end{pmatrix}, \; y^* = (\eta, b)^* = (-\eta^*, b^*). \tag{A.47}$$

This is clearly consistent with the guiding principle $(Mv)^* = M^* v^*$ because $(l.h.s.) = (r.h.s.) = \begin{pmatrix} a^*\eta^* + \beta^* b^* \\ -\gamma^*\eta^* + d^* b^* \end{pmatrix}$.

We have following properties which relates the transpose, hermitian conjugate and the complex conjugate.

(Prop) $(1)^T M = (M^*)^\dagger$, $(2) M^\dagger = {}^T(M^*)$.

(Proof) These properties can be verified by noting that the hermitian conjugate of the hermitian conjugate gives back the original quantity, which can be readily verified by definition.

1. $(M^*)^\dagger = (({}^T M)^\dagger)^\dagger = {}^T M$.
2. ${}^T(M^*) \overset{(1)}{=} ((M^*)^*)^\dagger = M^\dagger$. In the last equality, we have utilized the fact that, for a supermatrix, $(M^*)^* = M$, which can be readily verified from the explicit form of the complex conjugate (A.47).

This completes the proof of the above properties. (Q.E.D.)

Now we are ready to answer the question : *what do we mean by 'a supermatrix is real ?'*. In considering physics, we must take into account the reality condition. We utilize supermatrices in the context of expressing the action of superstring theory, the action must be real, and we are required to solidify the definition of the reality of a supermatrix.

(Def) A supermatrix M is real $\overset{def}{\Leftrightarrow}$ M is a mapping from *a real vector* to *a real vector*.

This statement is equivalent to, for the above definition of complex conjugate,

$$M^* = M. \tag{A.48}$$

[18] Be careful about the fact that ${}^T(M^\dagger)$ *is different from* $({}^T M)^\dagger = M^*$. They are computed to be ${}^T(M^\dagger) = \begin{pmatrix} a^* & -\beta^* \\ \gamma^* & d^* \end{pmatrix}$, ${}^T(v^\dagger) = \begin{pmatrix} -\eta^* \\ b^* \end{pmatrix}$ and ${}^T(y^\dagger) = (\eta^*, b^*)$ and these are *not* the complex conjugate of the vectors or the matrices.

This can be verified by noting the starting guiding principle that M is designed to satisfy $(Mv)^* = M^*v^*$. If $M^* = M$ is satisfied, $(Mv)^*$ is a real vector if v is real, because

$$(Mv)^* = M^*v^* \overset{v \text{ is real}}{=} M^*v \overset{M^*=M}{=} Mv. \tag{A.49}$$

The relationship (A.48) tells us the conditions for the components of M to satisfy. Noting the explicit form of complex conjugate (A.47), we can derive $a^* = a$, $d^* = d$, $\beta^* = \beta$ and $\gamma^* = -\gamma$, id est,

- a, β, d should be real.
- γ should be pure imaginary.

B Calculation of the Seeley-de-Witt coefficients

This appendix is devoted to introducing the basic technique of computing the Seeley-de-Witt coefficients in the heat kernel expansion. There are a number of ways to compute the coefficients, and here we focus on the calculation based on the Campbell-Baker-Hausdorff formula. We consider the general elliptic differential operator

$$D^2 = -\left(g^{ij}(x)\frac{d}{dx^i}\frac{d}{dx^j} + A^i(x)\frac{d}{dx^i} + B(x)\right). \tag{B.1}$$

In this section, we consider the d-dimensional spacetime, and the indices $i, j, \cdots$ run over $i, j, \cdots = 1, 2, \cdots, d$. These indices are allocated for the curved spacetime.

We consider the trace of the large N matrices in terms of the heat kernel. The trace of the operators are expressed using the complete system as

$$Tre^{-\tau D^2} = \int d^dx \langle x|e^{-\tau D^2}|x\rangle, \tag{B.2}$$

where the bracket $|x\rangle$ and $\langle x|$ satisfies $\sum_x |x\rangle\langle x| = 1$. However, it is difficult to consider the trace of a general operator, and we regard the operator as the sum of the first term of (B.1) and the perturbation around it. This is a famous procedure, and the perturbation is expressed in terms of the Seeley-de-Witt coefficients.

It is well known that the Green function is computed to be

$$\langle x| \exp\left(\tau g^{ij}(y)\frac{d}{dx^i}\frac{d}{dx^j}\right)|y\rangle = \frac{e(y)}{(2\pi\tau)^{\frac{d}{2}}} \exp\left(-\frac{(x-y)^i(x-y)^j g_{ij}(y)}{4\tau}\right). \tag{B.3}$$

Therefore, its trace is easily derived as

$$Tr\left(\exp\left(\tau g^{ij}(y)\frac{d}{dx^i}\frac{d}{dx^j}\right)\right) = \int d^dx \langle x| \exp\left(\tau g^{ij}(y)\frac{d}{dx^i}\frac{d}{dx^j}\right)|x\rangle = \int d^dx \frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}. \tag{B.4}$$

On the other hand, the heat kernel expansion of the general elliptic operator (B.1) is expanded as

$$Tr(e^{-\tau D^2}) = \int d^dx \langle x|e^{-\tau D^2}|x\rangle = \int d^dx \frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}(a_0 + \tau a_1 + \tau^2 a_2 + \cdots). \tag{B.5}$$

These coefficients $a_0, a_1, a_2, \cdots$ are called "the Seeley-de-Witt coefficients". (B.4) trivially gives $a_0 = 1$. However, the other coefficients give nontrivial results. In the following, we derive the first order a_1 for the general elliptic operator (B.1).

To this end, we divide $-\tau D^2$ as $-\tau D^2 = X + Y$, where

$$X = \tau\left(g^{ij}(y)\frac{d}{dx^i}\frac{d}{dx^j}\right), \tag{B.6}$$

$$Y = \tau\left((g^{ij}(x) - g^{ij}(y))\frac{d}{dx^i}\frac{d}{dx^j} + A^i(x)\frac{d}{dx^i} + B(x)\right). \tag{B.7}$$

From now on, we compute the exponential $\exp(X+Y)$ via the Campbell-Baker-Hausdorff formula, while the computation is rather involved:

$$e^Ae^B = \exp\left(A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])+\cdots\right). \tag{B.8}$$

Since we know that $\langle x|e^X|y\rangle = \frac{e(y)}{(2\pi\tau)^{\frac{d}{2}}}\exp\left(-\frac{1}{4\tau}(x-y)^i(x-y)^jg_{ij}(y)\right)$, the quantity in question is computed as

$$\begin{aligned} e^{X+Y}e^{-X} &= \exp\left(Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X+Y,[X+Y,-X]]+[-X,[-X,X+Y]])+\cdots\right) \\ &= \exp\left(Y+\frac{1}{2}[X,Y]+\frac{1}{12}(2[X,[X,Y]]-[Y,[Y,X]])+\cdots\right) \\ &= 1+Y+\frac{1}{2}[X,Y]+\frac{1}{6}[X,[X,Y]]+\frac{1}{12}[Y,[X,Y]]+\cdots \\ &\quad +\frac{1}{2}(Y+\frac{1}{2}[X,Y]+\frac{1}{6}[X,[X,Y]]+\frac{1}{12}[Y,[X,Y]]+\cdots)^2+\cdots \\ &= 1+Y+\frac{1}{2}[X,Y]+\frac{1}{6}[X,[X,Y]]+\frac{1}{2}Y^2+\frac{1}{8}[X,Y]^2+\frac{1}{3}Y[X,Y]+\frac{1}{6}[X,Y]Y+\cdots. \end{aligned} \tag{B.9}$$

Before we enter the computation of the quantity $\langle x|e^{X+Y}|y\rangle$, we summarize the formula of the differentiation of e^X:

$$\frac{de^X}{dx^i} = -\frac{1}{2\tau}(x-y)^jg_{ij}(y)e^X, \tag{B.10}$$

$$\frac{d^2e^X}{dx^{i_1}dx^{i_2}} = \left(-\frac{1}{2\tau}g_{i_1i_2}(y)+\frac{1}{4\tau^2}(x-y)^{l_1}(x-y)^{l_2}g_{i_1l_1}(y)g_{i_2l_2}(y)\right)e^X, \tag{B.11}$$

$$\begin{aligned} \frac{d^3e^X}{dx^{i_1}dx^{i_2}dx^{i_3}} &= \left(\frac{1}{4\tau^2}(x-y)^l(g_{i_1i_2}(y)g_{i_3l}(y)+g_{i_2i_3}(y)g_{i_1l}(y)+g_{i_3i_1}(y)g_{i_2l}(y))\right. \\ &\left.-\frac{1}{8\tau^3}(x-y)^{l_1}(x-y)^{l_2}(x-y)^{l_3}g_{i_1l_1}(y)g_{i_2l_2}(y)g_{i_3l_3}(y)\right)e^X, \end{aligned} \tag{B.12}$$

$$\begin{aligned} \frac{d^4e^X}{dx^{i_1}dx^{i_2}dx^{i_3}dx^{i_4}} &= \left(\frac{1}{4\tau^2}(g_{i_1i_2}(y)g_{i_3i_4}(y)+g_{i_2i_3}(y)g_{i_4i_1}(y)+g_{i_1i_3}(y)g_{i_2i_4}(y))\right. \\ &-\frac{1}{8\tau^3}(x-y)^{l_1}(x-y)^{l_2}(g_{i_1i_2}(y)g_{i_3l_1}(y)g_{i_4l_2}(y)+g_{i_2i_3}(y)g_{i_1l_1}(y)g_{i_4l_2}(y)+g_{i_1i_3}(y)g_{i_2l_1}(y)g_{i_4l_2}(y) \\ &+g_{i_1i_4}(y)g_{i_2l_1}(y)g_{i_3l_2}(y)+g_{i_2i_4}(y)g_{i_1l_1}(y)g_{i_3l_2}(y)+g_{i_3i_4}(y)g_{i_1l_1}(y)g_{i_2l_2}(y)) \\ &\left.+\frac{1}{16\tau^4}(x-y)^{l_1}(x-y)^{l_2}(x-y)^{l_3}(x-y)^{l_4}g_{i_1l_1}(y)g_{i_2l_2}(y)g_{i_3l_3}(y)g_{i_4l_4}(y)\right)e^X. \end{aligned} \tag{B.13}$$

Computation of Ye^X

We start with the computation of the easiest case:

$$\begin{aligned} Ye^X &= \tau\left((g^{ij}(x)-g^{ij}(y))\frac{d}{dx^i}\frac{d}{dx^j}+A^i(x)\frac{d}{dx^i}+B(x)\right)e^X \\ &= \tau\left((g^{ij}(x)-g^{ij}(y))(-\frac{1}{2}g_{ij}(y)+\frac{1}{4\tau}(x-y)^{l_1}(x-y)^{l_2}g_{il_1}(y)g_{jl_2}(y))\right. \\ &\quad\left.+B(x)-\frac{1}{2}A^i(x-y)^jg_{ij}(y)\right)e^X. \end{aligned} \tag{B.14}$$

Therefore, the trace is obtained by

$$Tr(Ye^X) = \int d^dx\langle x|Ye^X|x\rangle = \int d^dx\frac{\tau e(x)}{(2\pi\tau)^{\frac{d}{2}}}B(x). \tag{B.15}$$

Computation of $\frac{1}{2}[X,Y]e^X$

We next go on to a bit more complicated case, and we compute the operator $[X,Y]$ itself:

$$\begin{aligned}
[X,Y] &= \tau^2\left(g^{i_1i_2}(y)\frac{d}{dx^{i_1}}\frac{d}{dx^{i_2}}\right)\times\left((g^{j_1j_2}(x)-g^{j_1j_2}(y))\frac{d}{dx^{j_1}}\frac{d}{dx^{j_2}}+A^j(x)\frac{d}{dx^j}+B(x)\right)\\
&- \tau^2\left((g^{j_1j_2}(x)-g^{j_1j_2}(y))\frac{d}{dx^{j_1}}\frac{d}{dx^{j_2}}+A^j(x)\frac{d}{dx^j}+B(x)\right)\times\left(g^{i_1i_2}(y)\frac{d}{dx^{i_1}}\frac{d}{dx^{i_2}}\right)\\
&= \tau^2\left(2g^{i_1i_2}(y)(\frac{dg^{j_1j_2}(x)}{dx^{i_1}})\frac{d^3}{dx^{i_2}dx^{j_1}dx^{j_2}}+g^{i_1i_2}(y)(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}})\frac{d^2}{dx^{j_1}dx^{j_2}}\right.\\
&\quad +2g^{i_1i_2}(y)(\frac{dA^j(x)}{dx^{i_1}})\frac{d^2}{dx^{i_2}dx^j}+g^{i_1i_2}(y)(\frac{dA^j(x)}{dx^{i_1}dx^{i_2}})\frac{d}{dx^j}\\
&\quad \left.+2g^{i_1i_2}(y)(\frac{dB(x)}{dx^{i_1}})\frac{d}{dx^{i_2}}+g^{i_1i_2}(y)(\frac{d^2B(x)}{dx^{i_1}dx^{i_2}})\right). \qquad \text{(B.16)}
\end{aligned}$$

Therefore, the trace is computed to be, with the help of the formulae (B.13),

$$\begin{aligned}
&Tr(\frac{1}{2}[X,Y]e^X)=\int d^dx\langle x|\frac{1}{2}[X,Y]e^X|x\rangle\\
=&\int d^dx\frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}\left\{\tau\left(-\frac{1}{4}g^{i_1i_2}(x)g_{j_1j_2}(x)(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}})-\frac{1}{2}(\frac{dA^i(x)}{dx^i})\right)+\frac{\tau^2}{2}g^{i_1i_2}(x)(\frac{d^2B(x)}{dx^{i_1}dx^{i_2}})\right\}.
\end{aligned}$$

(B.17)

Computation of $\frac{1}{6}[X,[X,Y]]e^X$

We compute the operator $[X,[X,Y]]$ as

$$\begin{aligned}
&[X,[X,Y]]=\tau^3\left(4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{k_1}})\frac{d^4}{dx^{i_2}dx^{k_2}dx^{j_1}dx^{j_2}}\right.\\
&+4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^3g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}dx^{k_1}})\frac{d^3}{dx^{k_2}dx^{j_1}dx^{j_2}}\\
&+g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^4g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}dx^{k_1}dx^{k_2}})\frac{d^2}{dx^{j_1}dx^{j_2}}+4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^2A^j(x)}{dx^{i_1}dx^{k_1}})\frac{d^3}{dx^{i_2}dx^{k_2}dx^j}\\
&+4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^3A^j(x)}{dx^{i_1}dx^{i_2}dx^{k_1}})\frac{d^2}{dx^{i_2}dx^j}+g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^4A^j(x)}{dx^{i_1}dx^{i_2}dx^{k_1}dx^{k_2}})\frac{d}{dx^j}\\
&+4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^2B(x)}{dx^{i_1}dx^{k_1}})\frac{d^2}{dx^{i_2}dx^{k_2}}+4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^3B(x)}{dx^{i_1}dx^{i_2}dx^{k_1}})\frac{d}{dx^{k_2}}\\
&\left.+g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^4B(x)}{dx^{i_1}dx^{i_2}dx^{k_1}dx^{k_2}})\right). \qquad \text{(B.18)}
\end{aligned}$$

Therefore, the trace is computed as

$$\begin{aligned}
&Tr(\frac{1}{6}[X,[X,Y]]e^X)=\int d^dx\langle x|\frac{1}{6}[X,[X,Y]]e^X|x\rangle\\
=&\int d^dx\frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}\left\{\tau\left(\frac{1}{6}g^{i_1i_2}(x)g_{j_1j_2}(x)(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}})+\frac{1}{3}(\frac{d^2g^{ij}(x)}{dx^idx^j})\right)\right.\\
&-\tau^2\left(\frac{1}{12}g^{i_1i_2}(x)g^{j_1j_2}(x)g^{k_1k_2}(x)(\frac{d^4g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}dx^{k_1}dx^{k_2}})\right.\\
&\left.\left.+\frac{1}{3}g^{i_1i_2}(x)(\frac{d^3A^j(x)}{dx^{i_1i_2j}})+\frac{1}{3}g^{i_1i_2}(x)(\frac{d^2B(x)}{dx^{i_1}dx^{i_2}})\right)+\frac{\tau^3}{6}(g^{i_1i_2}(x)g^{j_1j_2}(x))(\frac{d^4B(x)}{dx^{i_1}dx^{i_2}dx^{j_1}dx^{j_2}})\right\}.
\end{aligned}$$

(B.19)

Computation of $\frac{1}{2}Y^2e^X$

The next job is the computation of the term $\frac{1}{2}Y^2$:

$$Y^2 = \tau^2\left((g^{i_1i_2}(x)-g^{i_1i_2}(y))\frac{d^2}{dx^{i_1}dx^{i_2}}+A^i(x)\frac{d}{dx^i}+B(x)\right)$$

$$
\begin{aligned}
& \times\left((g^{j_1j_2}(x)-g^{j_1j_2}(y))\frac{d^2}{dx^{j_1}dx^{j_2}}+A^j(x)\frac{d}{dx^j}+B(x)\right)\\
= \quad & \tau^2\left((g^{i_1i_2}(x)-g^{i_1i_2}(y))(g^{j_1j_2}(x)-g^{j_1j_2}(y))\frac{d^4}{dx^{i_1}dx^{i_2}dx^{j_1}dx^{j_2}}\right.\\
& +2(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{dg^{j_1j_2}(x)}{dx^{i_1}})\frac{d^3}{dx^{i_2}dx^{j_1}dx^{j_2}}+(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}})\frac{d^2}{dx^{j_1}dx^{j_2}}\\
& +2(g^{i_1i_2}(x)-g^{i_1i_2}(y))A^j(x)\frac{d^3}{dx^{i_1}dx^{i_2}dx^j}+2(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{dA^j(x)}{dx^{i_1}})\frac{d^2}{dx^{i_2}dx^j}\\
& +(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{d^2A^j(x)}{dx^{i_1}dx^{i_2}})\frac{d}{dx^j}+(g^{i_1i_2}(x)-g^{i_1i_2}(y))B(x)\frac{d^2}{dx^{i_1}dx^{i_2}}\\
& +2(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{dB(x)}{dx^{i_1}})\frac{d}{dx^{i_2}}+(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{d^2B(x)}{dx^{i_1}dx^{i_2}})\\
& +A^i(x)(\frac{dg^{j_1j_2}(x)}{dx^i})\frac{d^2}{dx^{j_1}dx^{j_2}}+A^i(x)A^j(x)\frac{d^2}{dx^idx^j}+A^i(x)B(x)\frac{d}{dx^i}+A^i(x)(\frac{dB(x)}{dx^i})\\
& \left.+(g^{i_1i_2}(x)-g^{i_1i_2}(y))B(x)\frac{d^2}{dx^{j_1}dx^{j_2}}+B(x)A^i(x)\frac{d}{dx^i}+B(x)B(x)\right). \qquad \text{(B.20)}
\end{aligned}
$$

The trace is thus

$$
\begin{aligned}
& Tr(\frac{1}{2}Y^2e^X)=\int d^dx\langle x|\frac{1}{2}Y^2e^X|x\rangle\\
= \quad & \int d^dx\frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}\left\{\tau\left(-\frac{1}{4}A^i(x)g_{j_1j_2}(x)(\frac{dg^{j_1j_2}(x)}{dx^i})-\frac{1}{4}A^i(x)A^j(x)g_{ij}(x)\right)\right.\\
& \left.+\tau^2\left(\frac{1}{2}A^i(x)(\frac{dB(x)}{dx^i})+\frac{1}{2}B(x)B(x)\right)\right\}. \qquad \text{(B.21)}
\end{aligned}
$$

Computation of $\frac{1}{8}[X,Y]^2e^X$

We next compute the commutator $[X,Y]^2$. From now on, the computation becomes more complicated than before, and we give only the trace:

$$
\begin{aligned}
& Tr(\frac{1}{8}[X,Y]^2e^X)=\int d^dx\langle x|\frac{1}{8}[X,Y]^2e^X|x\rangle\\
= \quad & \int d^dx\frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}\left\{\tau\left(-\frac{1}{16}g^{ik}(x)g_{j_1j_2}(x)g_{l_1l_2}(x)(\frac{dg^{j_1j_2}(x)}{dx^i})(\frac{dg^{l_1l_2}(x)}{dx^k})\right.\right.\\
& -\frac{1}{8}g^{ik}(x)g_{j_1l_1}(x)g_{j_2l_2}(x)(\frac{dg^{j_1j_2}(x)}{dx^i})(\frac{dg^{l_1l_2}(x)}{dx^k})-\frac{1}{4}(\frac{dg^{j_1j_2}(x)}{dx^{j_1}})(\frac{dg^{l_1l_2}(x)}{dx^{j_2}})g_{l_1l_2}(x)\\
& \left.\left.-\frac{1}{4}g_{j_2l_2}(x)(\frac{dg^{l_1l_2}(x)}{dx^{j_1}})(\frac{dg^{j_1j_2}(x)}{dx^{l_1}})-\frac{1}{4}g_{ij}(x)(\frac{dg^{ip}(x)}{dx^p})(\frac{dg^{jq}(x)}{dx^q})\right)+\mathcal{O}(\tau^2)\right\}. \qquad \text{(B.22)}
\end{aligned}
$$

Computation of $\frac{1}{3}Y[X,Y]e^X$

$$
\begin{aligned}
& Tr(\frac{1}{3}Y[X,Y]e^X)=\int d^dx\langle x|\frac{1}{3}Y[X,Y]e^X|x\rangle\\
= \quad & \int d^dx\frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}\left\{\tau\left(\frac{1}{6}A^i(x)(\frac{dg^{j_1j_2}(x)}{dx^i})g_{j_1j_2}(x)+\frac{1}{3}A^i(x)g_{ij}(x)(\frac{dg^{j_1j_2}(x)}{dx^{j_2}})\right)\right.\\
& -\tau^2\left(\frac{1}{6}g^{k_1k_2}(x)g_{ij}(x)A^i(x)(\frac{d^2A^j(x)}{dx^{k_1}dx^{k_2}})+\frac{1}{3}A^i(x)(\frac{dB(x)}{dx^i})\right.\\
& \quad +\frac{1}{6}g^{k_1k_2}(x)g_{j_1j_2}(x)(\frac{d^2g^{j_1j_2}(x)}{dx^{k_1}dx^{k_2}})B(x)+\frac{1}{6}g^{k_1k_2}(x)g_{j_1j_2}(x)A^i(x)(\frac{d^3g^{j_1j_2}(x)}{dx^idx^{k_1}dx^{k_2}})\\
& \quad \left.+\frac{1}{3}A^i(x)(\frac{d^2A^j(x)}{dx^idx^j})+\frac{1}{3}B(x)(\frac{dA^i(x)}{dx^i})\right)\\
& \left.+\frac{\tau^3}{3}B(x)g^{k_1k_2}(x)(\frac{d^2B(x)}{dx^{k_1}dx^{k^2}})\right\}. \qquad \text{(B.23)}
\end{aligned}
$$

Computation of $\frac{1}{6}[X,Y]Ye^X$

$$
\begin{aligned}
&Tr(\frac{1}{6}[X,Y]Ye^X) = \int d^dx \langle x|\frac{1}{6}[X,Y]Ye^X|x\rangle \\
= &\int d^dx \frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}} \left\{ \tau \left(\frac{1}{12} g^{k_1k_2}(x) g_{i_1i_2}(x) g_{j_1j_2}(x) (\frac{dg^{i_1i_2}(x)}{dx^{k_1}})(\frac{dg^{j_1j_2}(x)}{dx^{k_2}}) \right.\right. \\
&+\frac{1}{6} g^{k_1k_2}(x) g_{i_1j_1}(x) g_{i_2j_2}(x) (\frac{dg^{i_1i_2}(x)}{dx^{k_1}})(\frac{dg^{j_1j_2}(x)}{dx^{k_2}}) \\
&+\frac{1}{6} g_{i_1i_2}(x) (\frac{dg^{i_1i_2}(x)}{dx^{j_1}})(\frac{dg^{j_1j_2}(x)}{dx^{j_2}}) + \frac{1}{3} g_{i_2j_2}(x) (\frac{dg^{j_1j_2}(x)}{dx^{i_1}})(\frac{dg^{i_1i_2}(x)}{dx^{j_1}}) \\
&\left.\left. +\frac{1}{12} g_{j_1j_2}(x) \; (\frac{dg^{j_1j_2}(x)}{dx^{i}}) A^i(x) + \frac{1}{6}(\frac{dg^{j_1j_2}(x)}{dx^{j_1}}) g_{j_2i}(x) A^i(x) \right) + \mathcal{O}(\tau^2) \right\}. \qquad \text{(B.24)}
\end{aligned}
$$

Summing up these terms altogether, we obtain the general form of the first-order coefficient $a_1(x)$ as

$$
\begin{aligned}
a_1(x) \;=\;& B(x) - \frac{1}{2}(\frac{dA^i(x)}{dx^i}) + \frac{1}{3}(\frac{d^2g^{ij}(x)}{dx^idx^j}) - \frac{1}{12} g^{i_1i_2}(x) g_{j_1j_2}(x) (\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}}) \\
+\;& \frac{1}{12} g_{i_2j_2}(x) (\frac{dg^{j_1j_2}(x)}{dx^{i_1}})(\frac{dg^{i_1i_2}(x)}{dx^{j_1}}) - \frac{1}{4} A^i(x) A^j(x) g_{ij}(x) + \frac{1}{2} A^i(x) g_{ij_1}(x) (\frac{dg^{j_1j_2}(x)}{dx^{j_2}}) \\
+\;& \frac{1}{48} g^{k_1k_2}(x) g_{i_1i_2}(x) g_{j_1j_2}(x) (\frac{dg^{i_1i_2}(x)}{dx^{k_1}})(\frac{dg^{j_1j_2}(x)}{dx^{k_2}}) \\
+\;& \frac{1}{24} g^{k_1k_2}(x) g_{i_1j_1}(x) g_{i_2j_2}(x) (\frac{dg^{i_1i_2}(x)}{dx^{k_1}})(\frac{dg^{j_1j_2}(x)}{dx^{k_2}}) \\
-\;& \frac{1}{12} g_{i_1i_2}(x) (\frac{dg^{i_1i_2}(x)}{dx^{j_1}})(\frac{dg^{j_1j_2}(x)}{dx^{j_2}}) - \frac{1}{4} g_{ij}(x) (\frac{dg^{ip}(x)}{dx^p})(\frac{dg^{jq}(x)}{dx^q}). \qquad \text{(B.25)}
\end{aligned}
$$

Especially, the Seeley-de-Witt coefficient of the Laplace Beltrami operator

$$
\begin{aligned}
\Delta(x) \;=\;& \frac{1}{\sqrt{g(x)}} \left(\frac{d}{dx^i} \sqrt{g(x)} g^{ij}(x) \frac{d}{dx^j} \right) \\
=\;& g^{ij}(x) \frac{d}{dx^i} \frac{d}{dx^j} + \left(\left(\frac{dg^{ij}(x)}{dx^j} \right) - \frac{1}{2} g^{ij}(x) \left(\frac{d}{dx^j} g^{kl}(x) \right) g_{kl}(x) \right) \frac{d}{dx^i} \qquad \text{(B.26)}
\end{aligned}
$$

is important on the practical ground. This amounts to $A^i(x) = \left(\frac{dg^{ij}(x)}{dx^j}\right) - \frac{1}{2} g^{ij}(x) \left(\frac{d}{dx^j} g^{kl}(x)\right) g_{kl}(x)$ and $B(x) = 0$, in the general form (B.1). Plugging this into the general formula of a_1 (B.25), we obtain the following important result:

$$
a_1(x) = \frac{R(x)}{6}, \qquad \text{(B.27)}
$$

where $R(x)$ is the Ricci scalar of the spacetime metric. This result is used in the context of the superstring theory in deriving the Weyl anomaly and the corresponding Liouville action.

Derivation of (4.23)

With the above idea in mind, let us derive the result (4.23) for the simplest example of the calculation in our case. We expand the differential operator $-\tau D^2$ given by (4.7) around $X = \tau \partial_\mu \partial^\mu$. Namely, we express $-\tau D^2$ as $-\tau D^2 = X + Y$, and perform a perturbative expansion around X. The heat kernel for X is given by $\langle x|e^X|y\rangle = \int \frac{d^dx}{(2\pi\tau)^{\frac{d}{2}}} \exp\left(-\frac{(x-y)_\mu (x-y)^\mu}{4\tau}\right)$. Here, we identify the field $a^{(i)}{}_\mu(x)$ with the vielbein, and we expand this around the flat background metric as $a^{(i)}{}_\mu(x) = (a^{(i)}{}_\mu)_0 + h^i{}_\mu(x)$, where $(a^{(i)}{}_\mu)_0 (a^{(j)}{}_\mu)_0 = \eta^{ij}$. Then, D is more explicitly expressed as

$$
\begin{aligned}
D \;=\;& \Gamma^\mu \left(a_\mu(x) + i\partial_\mu + \frac{i}{2}(\partial_i h^i{}_\mu(x)) + ih^i{}_\mu(x)\partial_i + \cdots \right) + \frac{i}{3!} \Gamma^{\mu_1\mu_2\mu_3} (a_{\mu_1\mu_2\mu_3}(x) + \cdots) \\
-\;& \frac{1}{5!} \Gamma^{\mu_1\cdots\mu_5} (a_{\mu_1\cdots\mu_5}(x) + \cdots) - \frac{i}{7!} \Gamma^{\mu_1\cdots\mu_7} (a_{\mu_1\cdots\mu_7}(x) + \cdots) + \frac{1}{9!} \Gamma^{\mu_1\cdots\mu_9} (a_{\mu_1\cdots\mu_9}(x) + \cdots).
\end{aligned}
$$

(B.28)

Now, we are interested in the mass term of the vector fields $a_{\mu_1\cdots\mu_{2n-1}}(x)$. In the Campbell-Baker-Hausdorff formula (B.9), only the terms Y and $\frac{1}{2}Y^2$ contribute to their mass terms. Now, we have a close look at these two contribution.

For Y, only the elements of the rank-0 gamma matrices contribute to the heat kernel, because the trace tr for the 32×32 gamma matrices projects out the higher-rank gamma matrices. Therefore, the contribution to the mass term is clearly

$$\begin{aligned}\langle x|trYe^X|x\rangle &= \int \frac{d^dx}{(2\pi\tau)^{\frac{d}{2}}} 32\tau \Big(-a_\mu(x)a^\mu(x) - \frac{1}{3!}a_{\mu_1\mu_2\mu_3}(x)a^{\mu_1\mu_2\mu_3}(x) - \frac{1}{5!}a_{\mu_1\cdots\mu_5}(x)a^{\mu_1\cdots\mu_5}(x) \\ &\quad -\frac{1}{7!}a_{\mu_1\cdots\mu_7}(x)a^{\mu_1\cdots\mu_7}(x) - \frac{1}{9!}a_{\mu_1\cdots\mu_9}(x)a^{\mu_1\cdots\mu_9}(x)\Big), \end{aligned} \tag{B.29}$$

where the constant 32 comes from the trace $tr\mathbf{1}_{32\times 32}$. We next compute the effect of $\frac{1}{2}Y^2$. This time, the higher-rank terms of Y contribute, because the product of the gamma matrices survives. The relevant terms are as follows:

$$\begin{aligned} Y &= \tau\Big(2ia_\mu(x)\partial_\mu - \frac{2}{2!}\Gamma^{\nu_1\nu_2}a_{\mu\nu_1\nu_2}(x)\partial_\mu - \frac{2i}{4!}\Gamma^{\nu_1\cdots\nu_4}a_{\mu\nu_1\cdots\nu_4}(x)\partial_\mu + \frac{2}{6!}\Gamma^{\nu_1\cdots\nu_6}a_{\mu\nu_1\cdots\nu_6}(x)\partial_\mu \\ &+ \frac{2i}{8!}\Gamma^{\nu_1\cdots\nu_8}a_{\mu\nu_1\cdots\nu_8}(x)\partial_\mu\Big) + \cdots, \end{aligned}$$

where $\cdots$ denotes the omission of the irrelevant terms. This gives the contribution

$$\begin{aligned} \langle x|\frac{1}{2}trY^2e^X|x\rangle &= \langle x|\frac{\tau^2}{2} \times 32 \times \Big(-4a^\mu(x)a^\nu(x) - \frac{4}{2!}a^\mu{}_{\rho_1\rho_2}(x)a^{\nu\rho_1\rho_2}(x) - \frac{4}{4!}a^\mu{}_{\rho_1\cdots\rho_4}(x)a^{\nu\rho_1\cdots\rho_4}(x) \\ &\quad -\frac{4}{6!}a^\mu{}_{\rho_1\cdots\rho_6}(x)a^{\nu\rho_1\cdots\rho_6}(x) - \frac{4}{8!}a^\mu{}_{\rho_1\cdots\rho_8}(x)a^{\nu\rho_1\cdots\rho_8}(x)\Big)\partial_\mu\partial_\nu e^X|x\rangle \\ &= \int\frac{d^dx}{(2\pi\tau)^{\frac{d}{2}}}32\tau\Big(a_\mu(x)a^\mu(x) + \frac{1}{2!}a_{\mu_1\mu_2\mu_3}(x)a^{\mu_1\mu_2\mu_3}(x) + \frac{1}{4!}a_{\mu_1\cdots\mu_5}(x)a^{\mu_1\cdots\mu_5}(x) \\ &\quad +\frac{1}{6!}a_{\mu_1\cdots\mu_7}(x)a^{\mu_1\cdots\mu_7}(x) + \frac{1}{8!}a_{\mu_1\cdots\mu_9}(x)a^{\mu_1\cdots\mu_9}(x)\Big), \end{aligned} \tag{B.30}$$

where we have utilized the formula (B.11) for $g_{i_1i_2}(y) = \eta_{i_1i_2}$. We obtain the mass term (4.23) in question by the sum of (B.29) and (B.30).

C Basic knowledge for the numerical simulation of the matrix model

This appendix is devoted to introducing the basic knowledge of the Monte Carlo simulation of the matrix model. While the following ideas can be readily applied to the QCD, the programming for the matrix model is much simpler than for the quantum field theory.

C.1 Review of the Monte Carlo simulation

The Monte Carlo simulation plays an extremely important role in the numerical simulation. In many cases, the analytical computation may be too complicated to handle. Let us think about the four-dimensional 10,000-site lattice gauge theory. This system has 40,000 sites. Therefore, even for the simplest Z_2 gauge theory, we have to compute the sum of the $2^{40,000} \sim 1.58 \times 10^{12041}$ configurations. The analytical computation of the partition function is almost impossible and this leads us to resort to the numerical treatment.

To put it simply, the fundamental idea of the Monte Carlo simulation is to produce the equilibrium ensemble artificially. In the case of the matrix model, the state C is identified with the element of the $N \times N$ matrices; namely

$$C = \{(A_\mu)_{ij}, \cdots, \}. \tag{C.1}$$

For the quantum field theory, C of course corresponds to the gauge field configuration $U^{ij}_\mu(x)$. Since the sequential integral is represented as $\int dA_\mu = \sum_C \cdots$, the partition function is rewritten as $Z = \sum_C \exp(-S(C))$. In the Monte Carlo simulation, we derive the configuration C_k that complies with the Boltzmann probability

$$P_{\text{bol}}(C_k) = Z^{-1} \exp(-S(C_k)). \tag{C.2}$$

To this end, we consider the following Markov process. Namely, the probability to transform from the configuration C_{k-1} to the new one C_k depends only on the state C_{k-1} and C_k. This probability never depends on the previous history $C_1, C_2, \cdots, C_{k-2}$. In this sense, this probability is denoted as

$$P(C_{k-1} \to C_k) = P(C_{k-1}, C_k), \tag{C.3}$$

since the probability is the function of only C_{k-1} and C_k. The following "detailed balance condition" is a vital constraint on the Boltzmann distribution:

$$e^{-S(C)} P(C, C') = e^{-S(C')} P(C', C). \tag{C.4}$$

For the transformation complying with the detailed balance condition, we have the following two crucial properties:

1. An equilibrium sequence of the state transforms into another equilibrium state.
2. A nonequilibrium sequence approaches to the equilibrium state.

It is easy to prove the first statement. While it is trivial, we recall the following properties of the probabilities. Firstly, $\sum_C P(C, C') = \sum_{C'} P(C, C') = 1$. Secondly, when E' is obtained from E through the Markov process and P' is the probability for the ensemble E', we have $P'(C) = \sum_{C'} P(C, C')P(C')$. The first property is verified by taking the sum of the both hand sides of (C.4) with respect to C. Now, the previous state $P_{\text{bol}}(C)$ complies with the Boltzmann distribution $P_{\text{bol}}(C) = Z^{-1}e^{-S(C)}$. Then, we have

$$P'(C') = \sum_C P_{\text{bol}}(C)P(C, C') = \sum_C P_{\text{bol}}(C')P(C', C) = Z^{-1}e^{-S(C')}. \tag{C.5}$$

Thus, the newly emerging state also complies with the Boltzmann distribution.

The second property is substantiated by defining the distance of the two ensembles E and E' as

$$|E - E'| = \sum_C |P(C) - P'(C)|. \tag{C.6}$$

Then, the distance from the Boltzmann distribution is given by

$$\begin{aligned} |E' - E_{\text{bol}}| &= \sum_C |P'(C) - P_{\text{bol}}(C)| = \sum_C \left| \sum_{C'} P(C, C')(P(C') - P_{\text{bol}}(C')) \right| \\ &\leq \sum_C \sum_{C'} P(C, C')|P(C') - P_{\text{bol}}(C')| = \sum_{C'} |P(C') - P_{\text{bol}}(C')| = |E - E_{\text{bol}}|. \end{aligned} \tag{C.7}$$

This implies that the distance from the Boltzmann state is smaller as we iterate the Markov process obeying the detailed balance condition.

There are two major algorithms that satisfy the detailed balance condition. One is the heat bath algorithm. The heat bath algorithm is used when we have a clear correspondence between the uniform random number and the configuration of the fields. In this case, the probability to transform to the configuration C' is given by

$$P(C, C') \propto e^{-S(C')}. \tag{C.8}$$

In this process, the probability $P(C, C')$ depends only on the new configuration, and does not even depend on the previous state. It is trivial that this satisfies the detailed balance condition. In analyzing the matrix model, we resort to the heat bath algorithm. We postpone the explicit example to the subsequent sections for the matrix model simulation.

The other is the Metropolis algorithm. Firstly, we choose the configuration C' at random. Then, we compute the difference $\Delta S = S(C') - S(C)$. We adopt the new state at the probability $\min(1, e^{-\Delta S})$. Namely, the probability is given by

$$P(C, C') \to \begin{cases} 1 & (\text{if } S(C) > S(C')), \\ e^{-\Delta S} = e^{-S(C')} e^{S(C)} & (\text{if } S(C) < S(C')). \end{cases} \tag{C.9}$$

We take a uniform random number $r \in [0, 1]$. When $r \leq e^{-\Delta S}$ we adopt the new configuration C', and otherwise we jettison the new configuration C' and keep the status quo. It is easy to verify that the Metropolis algorithm also satisfies the detailed balance condition, by noting that the probability for the inverse transformation is given by

$$P(C', C) \to \begin{cases} e^{\Delta S} = e^{S(C')} e^{-S(C)} & (\text{if } S(C) > S(C')), \\ 1 & (\text{if } S(C) < S(C')). \end{cases} \tag{C.10}$$

The advantage of the Metropolis algorithm is that it can be applied to more various systems. When it is impossible to build the heat bath algorithm, the Metropolis algorithm is here to stay. A good example is the supersymmetric matrix model. When the model has fermions, we often use the hybrid Monte Carlo simulation. It is impossible to handle the Grassmann numbers numerically. Therefore, if we are to treat the model equipped with the fermion, we must integrate out the fermion analytically. Then, we evaluate the Pfaffian composed of the bosonic fields with the numerical method.

C.2 Simulation of the quadratic matrix model

We first start with the simplest example of the matrix model. While it is a very simple toy model, we scrutinize this case in full detail because this is the heart of the heat bath algorithm of the more complicated matrix models. We deal with the following quadratic matrix model:

$$S = \frac{N}{2} Tr \phi^2. \tag{C.11}$$

Here, ϕ is an $N \times N$ hermitian matrix, and this model is invariant under the $U(N)$ unitary transformation. Its path integral is given by

$$Z = \int d^{N^2} \phi e^{-S} = \int d^{N^2} \phi \exp(-\frac{N}{2} Tr \phi^2). \tag{C.12}$$

Analytically, the propagator of this matrix model is evaluated as follows[19]:

$$\langle \phi_{ij} \phi_{kl} \rangle = \frac{1}{Z} \int d^{N^2} \phi \phi_{ij} \phi_{kl} \exp(-\frac{N}{2} Tr \phi^2) = \frac{1}{N} \delta_{il} \delta_{jk}. \tag{C.13}$$

The proof of this propagator goes as follows. We note that, due to the hermiticity of ϕ, the trace is written as

$$\frac{N}{2} Tr \phi^2 = \frac{N}{2} \sum_{i,j=1}^{N} \phi_{ij} \phi_{ji} = N \sum_{1 \leq i < j \leq N} \phi_{ij} \phi_{ij}^* + \frac{N}{2} \sum_{i=1}^{N} \phi_{ii} \phi_{ii}. \tag{C.14}$$

[19] The result (C.13) is applicable to the case $\phi \in U(N)$. Here, we touch on the propagator for the $SU(N)$ case. This can be derived from (C.13) by subtracting the trace part. Namely, we replace ϕ_{ii} with $\phi'_{ii} = \phi_{ii} - \frac{1}{N} \sum_{i=1}^{N} \phi_{ii}$. The propagator $\langle \phi'_{ii} \phi'_{ll} \rangle$ is computed as

$$\langle \phi'_{ii} \phi'_{ll} \rangle = \langle \phi_{ii} \phi_{ll} \rangle - 2 \frac{1}{N} \sum_{j=1}^{N} \langle \phi_{ii} \phi_{jj} \rangle + \frac{1}{N^2} \sum_{j,k=1}^{N} \langle \phi_{jj} \phi_{kk} \rangle = \frac{1}{N} \left(\delta_{il} - \frac{1}{N} \right).$$

Therefore, the Feynman rule is replaced for $SU(N)$ case by

$$\langle \phi_{ij} \phi_{kl} \rangle = \frac{1}{N} \left(\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right).$$

We separate ϕ_{ij} into the real and the imaginary part as

$$\phi_{ij} = \frac{X_{ij} + iY_{ij}}{\sqrt{2N}} (= \phi^*_{ji}). \tag{C.15}$$

Here, X_{ij} and Y_{ij} are real c-number. The diagonal parts are real numbers, and they are rewritten using the real numbers W_i as

$$\phi_{ii} = \frac{W_i}{\sqrt{N}}. \tag{C.16}$$

Then, the quadratic term is written as

$$\frac{N}{2} Tr\phi^2 = \frac{1}{2} \left(\sum_{i=1}^{N} W_i^2 + \sum_{1 \leq i < j \leq N} (X_{ij}^2 + Y_{ij}^2) \right). \tag{C.17}$$

The derivation of the propagator reduces to the simple Gaussian integral:

$$\frac{1}{a} = \frac{\int_{-\infty}^{+\infty} dx x^2 \exp(-\frac{ax^2}{2})}{\int_{-\infty}^{+\infty} dx \exp(-\frac{ax^2}{2})}. \tag{C.18}$$

- $\langle \phi_{ii} \phi_{ll} \rangle = \frac{1}{N} \langle W_i W_l \rangle$ survives only for $i = l$.
- For $\langle \phi_{ij} \phi_{kl} \rangle$ $(i \neq j)$, we note the following two results. Firstly, $\langle \phi_{ij} \phi_{ij} \rangle$ is shown to vanish as

$$\langle \phi_{ij} \phi_{ij} \rangle = \frac{1}{2N} \langle (\underbrace{X_{ij} X_{ij} - Y_{ij} Y_{ij}}_{\text{cancelled}} + 2i \underbrace{X_{ij} Y_{ij}}_{(*)}) \rangle = \frac{1-1}{2N} = 0. \tag{C.19}$$

  Here, (*) does not contribute ab initio, since this is a linear term of each X_{ij} and Y_{ij}. Secondly, we note that

$$\langle \phi_{ij} \phi_{ji} \rangle = \frac{1}{2N} \langle (X_{ij} X_{ij} + Y_{ij} Y_{ij}) \rangle = \frac{1}{N}. \tag{C.20}$$

  survives (namely when $i = l, j = k$).

This completes the proof of the Feynman rule (C.13).

Using the Feynman rule (C.13), we can easily derive the following vacuum expectation values:

$$\langle \frac{1}{N} Tr\phi^2 \rangle = 1, \;\; \langle \frac{1}{N} Tr\phi^4 \rangle = 2 + \frac{1}{N^2}, \;\; \langle (\frac{1}{N} Tr\phi^2)^2 \rangle = 1 + \frac{2}{N^2}. \tag{C.21}$$

We next go into the numerical treatment of this matrix model, by means of the heat bath algorithm. The rewriting of the action (C.17) plays a crucial role in the analysis. The advantage is that all W_i, X_{ij} and Y_{ij} are decoupled. Therefore, in the heat bath algorithm, we can update each of these real numbers independently. Recall that, in the heat bath algorithm (C.8), the probability for the new configuration depends only on the new configuration. Now, the partition function is rewritten as

$$Z = \int \prod_{i=1}^{N} dW_i \prod_{1 \leq i < j \leq N} dX_{ij} dY_{ij} \exp \left(-\frac{1}{2} \sum_{i=1}^{N} W_i^2 - \frac{1}{2} \sum_{1 \leq i < j \leq N} (X_{ij}^2 + Y_{ij}^2) \right). \tag{C.22}$$

Thus, for example, the new configuration of W_i complies with the following probability distribution.

$$P(W_i) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{W_i^2}{2}), \tag{C.23}$$

where $P(W_i)$ is normalized so that $\int_{-\infty}^{+\infty} dW_i P(W_i) = 1$. Namely, the updating of the quantities W_i, X_{ij}, Y_{ij} all reduces to the Gaussian distribution. The process of updating the matrix elements can be divided into the following three steps.

Generation of the uniform random number

Firstly, we introduce the numerical method to generate the sequence of the pseudo-random number. A simple way is the "congruence method". We generate the sequence of the random numbers $\{z_i\}$ by the following recursive formula:

$$z_{i+1} = az_i + c \pmod{2^{31} - 1}. \tag{C.24}$$

The initial term z_1 should be given by hand as a random seed. When we normalize this sequence as $\{\frac{z_i}{2^{31}-1}\}$, we obtain a uniform pseudo-random number [0,1]. Especially, it is known that we obtain a sensible pseudo-random sequence by taking $a = 5^{11}$ and $c = 0$. The drawback of the congruence method is the price of its simplicity. This random sequence has too short a cycle. However, this is not so problematic so long as we undergo the simulation of the matrix model, while this is a serious problem for the QCD simulation.

Generation of the Gaussian distribution

We next introduce the way to generate the Gaussian distribution from the uniform random numbers, generated in the previous step. We take two independent uniform random numbers $x, y \in [0,1]$. Then, we introduce the quantity r as

$$r = \sqrt{-a^2 \log x^2}. \tag{C.25}$$

The probability distribution of r is given by

$$P(r)dr = P(x)\frac{dx}{dr}dr = \frac{2r}{a^2}\exp(-(\frac{r}{a})^2). \tag{C.26}$$

We next introduce the quantities X, Y as

$$X = r\cos(2\pi y), \quad Y = r\sin(2\pi y). \tag{C.27}$$

We discern that this complies with the Gaussian distribution as

$$P(r)drdy \propto \exp\left(-\frac{1}{a^2}(X^2+Y^2)\right)dXdY. \tag{C.28}$$

Especially when $a = 1$, X and Y obey the normal Gaussian distribution.

Updating of the matrix elements

Now that we introduce the random number complying the Gaussian distribution, we update each of the matrix elements W_i, X_{ij} and Y_{ij} by the Gaussian distribution. When we finish updating all of them once, this means that we have completed one sweep. Then, we reiterate the sweeps until the system is sufficiently thermalized.

C.3 Simulation of the quartic matrix model

We next investigate the ϕ^4 matrix model. This model has a lot of lessons in treating the IIB matrix model and its extensions. The action is defined by

$$S = N\left(\frac{1}{2}Tr\phi^2 - \frac{g}{4}Tr\phi^4\right). \tag{C.29}$$

In order to perform the heat bath algorithm of this matrix model, we introduce the auxiliary field Q as

$$\tilde{S} = \frac{N}{2}Tr\phi^2 + \frac{N}{2}TrQ^2 - \alpha N Tr(Q\phi^2). \tag{C.30}$$

Here, $\alpha = \sqrt{\frac{g}{2}}$ and the auxiliary field Q is an $N \times N$ hermitian matrix. It is easy to verify that the action (C.30) reduces to (C.29) by integrating out Q:

$$\tilde{S} = \frac{N}{2}Tr(Q - \alpha\phi^2)^2 + S.$$

The idea to update the quantities Q and ϕ is similar to that of the quadratic case, and we proceed rather quickly. We rewrite the elements of Q as

$$Q_{ii} = \frac{W_i}{\sqrt{N}} + \alpha(\phi^2)_{ii}, \quad Q_{ij} = \frac{X_{ij} + iY_{ij}}{\sqrt{2N}} + \alpha(\phi^2)_{ij}. \tag{C.31}$$

In this way, the action $\tilde{S}$ is written as

$$\tilde{S} = \frac{1}{2}\left(\sum_{i=1}^{N} W_i^2 + \sum_{1\le i<j\le N} (X_{ij}^2 + Y_{ij}^2)\right) + S.$$

In the following, W_i $(i = 1, \cdots, N)$, X_{ij} and Y_{ij} $(1 \le i < j \le N)$ denote the real c-numbers that obey the normal Gaussian distribution.

We next update the elements of the matrix ϕ. To this end, we extract the dependence of $\tilde{S}$ on each component ϕ_{ij}. Firstly, the dependence on the diagonal part ϕ_{ii} is extracted as follows. In the following (up to (C.47)), we sometimes do not take a summation with respect to the duplicate indices.

$$\begin{aligned}
\tilde{S} &= \frac{N}{2}(\phi_{ii})^2 - \alpha N Q_{ii}(\phi_{ii})^2 - \alpha N \phi_{ii} \sum_{j\neq i}(\phi_{ij}Q_{ji} + Q_{ij}\phi_{ji}) + \cdots \\
&= \frac{Nc_i}{2}(\phi_{ii} - \frac{h_i}{c_i})^2 - \frac{N}{2}\frac{(h_i)^2}{c_i} + \cdots, \text{ where} \\
c_i &= 1 - 2\alpha Q_{ii}, \quad h_i = \alpha \sum_{j\neq i}(\phi_{ij}Q_{ji} + Q_{ij}\phi_{ji}).
\end{aligned} \tag{C.32}$$

The $\cdots$ denotes the terms independent of ϕ_{ii}. We omit them because these do not concern the updating of ϕ_{ii} Therefore, the diagonal parts ϕ_{ii} are updated as

$$\phi_{ii} = \frac{W_i}{\sqrt{Nc_i}} + \frac{h_i}{c_i}. \tag{C.33}$$

We next go on to the updating of ϕ_{ij}, whose dependence is extracted as

$$\begin{aligned}
\tilde{S} &= N|\phi_{ij}|^2 - \alpha N|\phi_{ij}|^2(Q_{ii} + Q_{jj}) - \left[\alpha N \phi_{ij}\left(\sum_{k\neq i}\phi_{jk}Q_{ki} + \sum_{k\neq j} Q_{jk}\phi_{ki}\right) + \text{c.c.}\right] + \cdots \\
&= Nc_{ij}\left|\phi_{ij} - \frac{h_{ij}}{c_{ij}}\right|^2 - N\frac{|h_{ij}|^2}{c_{ij}} + \cdots, \text{ where} \\
c_{ij} &= 1 - \alpha(Q_{ii} + Q_{jj}), \quad h_{ij} = \alpha\left(\sum_{k\neq j}\phi_{ik}Q_{kj} + \sum_{k\neq i} Q_{jk}\phi_{kj}\right).
\end{aligned} \tag{C.34}$$

This leads us to update the nondiagonal parts ϕ_{ij} as

$$\phi_{ij} = \frac{X_{ij} + iY_{ij}}{\sqrt{2Nc_{ij}}} + \frac{h_{ij}}{c_{ij}}. \tag{C.35}$$

This completes the algorithm of updating the matrix elements. However, we must recall that this matrix model is unbounded below. The eigenvalues near the origin is stable only for the large-N limit, and only *metastable* for the finite N. The function $V(x) = N(\frac{1}{2}x^2 - \frac{g}{4}x^4)$ has extrema at $x = \pm\frac{1}{\sqrt{g}}$, at which $V(\pm\frac{1}{\sqrt{g}}) = \frac{N}{4g}$. Therefore, the origin is barriered by the potential with the height $\frac{N}{4g}$. For the large-N limit, this barrier prevents the eigenvalues from dissipating outside the potential, and the system is kept finite. In [61], the following vacuum expectation value is computed analytically in the large-N limit for $0 \le g \le \frac{1}{12}$:

$$\lim_{N\to\infty}\langle\frac{1}{N}Tr\phi^2\rangle = \frac{a^2(4-a^2)}{3}, \text{ where } a^2 = \frac{2}{1+\sqrt{1-12g}}. \tag{C.36}$$

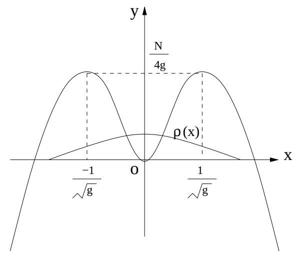

Figure 21: The metastability of the origin for finite N. The height of the potential barrier is $\frac{N}{4g}$.

Here, the parameter a gives the support of the eigenvalue distribution. Namely, the eigenvalues are distributed only in the region $[-2a, +2a]$. More accurately, the eigenvalue distribution is also derived as

$$\rho(\lambda) = \frac{1}{\pi}\sqrt{4a^2 - \lambda^2}\left(-\frac{1}{2}g\lambda^2 - ga^2 + \frac{1}{2}\right), \tag{C.37}$$

where $\rho(\lambda)$ is normalized as $\int_{-\infty}^{+\infty}\rho(\lambda)d\lambda = 1$. We delegate the detailed derivation to [61].

However, for the finite N, the height of the potential barrier is also finite, thus we see the divergence in the course of updating the eigenvalues. Namely, when the eigenvalues surpass the potential barrier, we

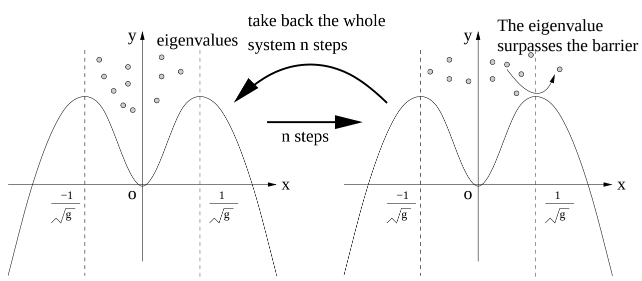

Figure 22: The reguralization trick utilized in [64]. When any one of the eigenvalue surpasses the potential barrier, we take back the whole system several steps.

see an avalanche of the eigenvalues rolling off and off the potential. This is what happens in the course of the divergence of the system. In order to evade the divergence, we need some regularization. Here, we utilize the same regularization trick as for the research of the Weingarten model [64]. Namely, when any one of the eigenvalues surpasses the potential barrier $\frac{1}{\sqrt{g}}$, we take back the whole system several n steps.

In explicitly writing our code, we store the history of several previous steps for the matrix elements of ϕ. When the eigenvalues happen to cordon the potential barrier, we recall the record of the history, and take back several steps (practically, around $n = 3, 4$ steps). In this way, we evade the divergence for finite N. Here, we plot the numerical result based on the heat bath algorithm for the quantity $\langle \frac{1}{N}Tr\phi^2\rangle$ for $N = 32$.

C.4 Simulation of the bosonic IIB matrix model with the Chern-Simons term

We next introduce the algorithm for the IIB matrix model with the Chern-Simons term. In this section, we define the action as

$$S = -\frac{N}{4}Tr[A_\mu, A_\nu]^2 - \frac{\lambda}{2k+1}N\epsilon_{\mu_1\cdots\mu_{2k+1}}TrA_{\mu_1}A_{\mu_2}\cdots A_{\mu_{2k+1}}. \tag{C.38}$$

Namely, we take the g in the action (2.111) to be $\frac{1}{g^2} = N$. When we set $\lambda = 0$, this is of course identical to the bosonic IIB matrix model, which has been scrutinized in [13]. There is not so big a difference from the bosonic IIB matrix model, because the Chern-Simons term is nothing but a linear term with respect to each A_μ.

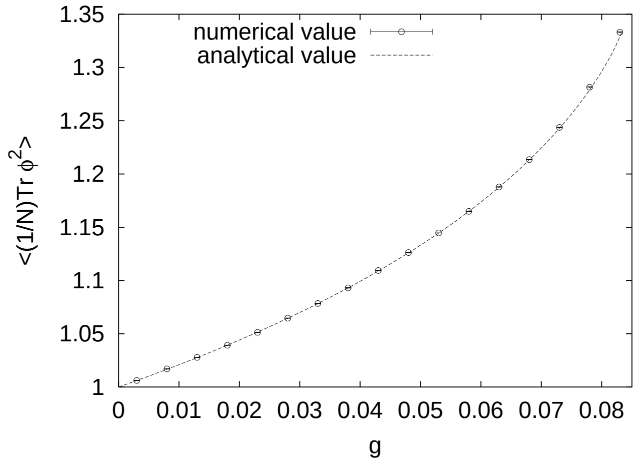

Figure 23: The plot of $\langle \frac{1}{N} Tr\phi^2 \rangle$ for the ϕ^4 matrix model for $N = 32$.

We define this matrix model in the $d = (2k+1)$-dimensional Euclidean space, and the indices $\mu, \nu, \cdots$ run over $1, 2, \cdots, d = (2k+1)$. The following argument is applicable to the bosonic IIB matrix model without the Chern-Simons term (namely when $\lambda = 0$) for an arbitrary dimensions $d \geq 3$(the even d is also acceptable)[20]. For the bosonic IIB matrix model without the Chern-Simons term, the readers are to ignore the Chern-Simons term in the following argument.

The action (C.38) is also analyzed via the heat bath algorithm. Firstly, we note that the quartic commutator in (C.38) is rewritten as

$$\begin{aligned} -\frac{N}{4}\sum_{\mu,\nu=1}^{d} Tr[A_\mu, A_\nu]^2 &= -\frac{N}{2}\sum_{1\leq\mu<\nu\leq d} Tr[A_\mu, A_\nu]^2 = N \sum_{1\leq\mu<\nu\leq d} [Tr(A_\mu^2 A_\nu^2) - Tr(A_\mu A_\nu A_\mu A_\nu)] \\ &= -\frac{N}{2}\sum_{1\leq\mu<\nu\leq d} TrG_{\mu\nu}^2 + 2N \sum_{1\leq\mu<\nu\leq d} Tr(A_\mu^2 A_\nu^2), \end{aligned} \tag{C.39}$$

where $G_{\mu\nu} = \{A_\mu, A_\nu\}$, and these are hermitian matrices because these are anti-commutators of A_μ. This leads us to introduce the auxiliary fields $Q_{\mu\nu}$ as

$$\tilde{S} = N \sum_{1\leq\mu<\nu\leq d} \left(\frac{1}{2} TrQ_{\mu\nu}^2 - Tr(Q_{\mu\nu}G_{\mu\nu}) + 2Tr(A_\mu^2 A_\nu^2) - \frac{\lambda}{2k+1} N\epsilon_{\mu_1\cdots\mu_{2k+1}} TrA_{\mu_1}A_{\mu_2}\cdots A_{\mu_{2k+1}} \right). \tag{C.40}$$

Here, $Q_{\mu\nu}$ are hermitian matrices, and satisfy $Q_{\mu\nu} = Q_{\nu\mu}$. $Q_{\mu\nu}$ is defined only for $\mu \neq \nu$. Of course, the action (C.40) is equivalent to (C.38) after we integrate out $Q_{\mu\nu}$:

$$\tilde{S} = \frac{N}{2} \sum_{1\leq\mu<\nu\leq d} Tr(Q_{\mu\nu} - G_{\mu\nu})^2 + S.$$

[20]The path integral of the two-dimensional bosonic IIB matrix model (without the Chern-Simons term)

$$S = -\frac{N}{4}\sum_{\mu,\nu=1}^{2} Tr[A_\mu, A_\nu]^2 = -\frac{N}{2} Tr[A_1, A_2]^2$$

is easily shown to diverge [24] for any size of the matrices N. When we diagonalize A_1 as $A_1 = \mathrm{diag}(\lambda_1, \lambda_2, \cdots \lambda_N)$, the diagonal parts of A_2 clearly do not contribute to the action. However, in the measure of the path integral

$$Z = \int dA_1 dA_2 e^{-S},$$

the integral for the diagonal parts of A_2 runs vacuously. This causes the divergence of the path integral.

Therefore, the matrices $Q_{\mu\nu}$ are updated as

$$(Q_{\mu\nu})_{ii} = \frac{W_i}{\sqrt{N}} + (G_{\mu\nu})_{ii}, \quad (Q_{\mu\nu})_{ij} = \frac{X_{ij} + iY_{ij}}{\sqrt{2N}} + (G_{\mu\nu})_{ij}. \tag{C.41}$$

We next update the matrices A_ρ. To this end, we extract the dependence of the action (C.40) on each component of $(A_\rho)_{ij}$. To this end, we rewrite the action (C.40) as

$$\begin{aligned}
\tilde{S} &= -N\sum_{\rho<\nu} TrQ_{\rho\nu}(A_\rho A_\nu + A_\nu A_\rho) - N\sum_{\mu<\rho} TrQ_{\mu\rho}(A_\mu A_\rho + A_\rho A_\mu) \\
&+ 2N\sum_{\rho<\nu} Tr(A_\rho^2 A_\nu^2) + 2N\sum_{\mu<\rho} Tr(A_\mu^2 A_\rho^2) \\
&- \frac{\lambda}{2k+1} N \sum_{\nu_i\neq\rho} \epsilon_{\nu_1\cdots\nu_{2k}\rho} TrA_{\nu_1}\cdots A_{\nu_{2k}}A_\rho + \cdots \\
&= -N\sum_{\mu\neq\rho} TrQ_{\mu\rho}(A_\mu A_\rho + A_\rho A_\mu) + 2N\sum_{\mu\neq\rho} Tr(A_\mu^2 A_\rho^2) - \lambda N \sum_{\nu_i\neq\rho} \epsilon_{\nu_1\cdots\nu_{2k}} TrA_{\nu_1}\cdots A_{\nu_{2k}}A_\rho + \cdots \\
&= -NTr(T_\rho A_\rho) + 2NTr(S_\rho A_\rho^2) + \cdots, \text{ where} && \text{(C.42)} \\
S_\rho &= \sum_{\mu\neq\rho} A_\mu^2, \quad T_\rho = \sum_{\mu\neq\rho}(A_\mu Q_{\rho\mu} + Q_{\rho\mu}A_\mu) + \lambda \sum_{\nu_i\neq\rho} \epsilon_{\nu_1\cdots\nu_{2k}} TrA_{\nu_1}\cdots A_{\nu_{2k}}. && \text{(C.43)}
\end{aligned}$$

The difference between the pure bosonic IIB matrix model and the model with the Chern-Simons term only comes in the definition of T_ρ. Otherwise, both cases go totally in the same way.

Using the rewriting (C.42), we first update the diagonal part $(A_\rho)_{ii}$. (C.42) is further rewritten as

$$\begin{aligned}
\tilde{S} &= -N(T_\rho)_{ii}(A_\rho)_{ii} + 2N(S_\rho)_{ii}(A_\rho)_{ii}^2 + 2N\sum_{j\neq i}[(S_\rho)_{ji}(A_\rho)_{ii}(A_\rho)_{ij} + (S_\rho)_{ij}(A_\rho)_{ji}(A_\rho)_{ii}] + \cdots \\
&= 2N(S_\rho)_{ii}(A_\rho)_{ii}^2 - 4Nh_i + \cdots = 2N(S_\rho)_{ii}\left((A_\rho)_{ii} - \frac{h_i}{(S_\rho)_{ii}}\right) - 2N\frac{h_i^2}{(S_\rho)_{ii}} + \cdots, \text{ where} && \text{(C.44)} \\
h_i &= \frac{1}{4}\left((T_\rho)_{ii} - 2\sum_{j\neq i}[(S_\rho)_{ji}(A_\rho)_{ij} + (S_\rho)_{ij}(A_\rho)_{ji}]\right).
\end{aligned}$$

Therefore, the diagonal parts $(A_\rho)_{ii}$ are updated as

$$(A_\rho)_{ii} = \frac{W_i}{\sqrt{4N(S_\rho)_{ii}}} + \frac{h_i}{(S_\rho)_{ii}}. \tag{C.45}$$

We next update the nondiagonal parts $(A_\rho)_{ij}$, whose dependence is now extracted as

$$\begin{aligned}
\tilde{S} &= -N[(T_\rho)_{ji}(A_\rho)_{ij} + (T_\rho)_{ij}(A_\rho)_{ji}] + 2N[(S_\rho)_{ii} + (S_\rho)_{jj}]|(A_\rho)_{ij}|^2 \\
&+ 2N\left[\sum_{k\neq j}(S_\rho)_{jk}(A_\rho)_{ki}(A_\rho)_{ij} + \sum_{k\neq i}(S_\rho)_{ik}(A_\rho)_{kj}(A_\rho)_{ji}\right. \\
&\left. + \sum_{k\neq j}(S_\rho)_{kj}(A_\rho)_{ji}(A_\rho)_{ik} + \sum_{k\neq i}(S_\rho)_{ki}(A_\rho)_{ij}(A_\rho)_{jk}\right] + \cdots \\
&= 2Nc_{ij}|(A_\rho)_{ij}|^2 - 2N(A_\rho)_{ij}h_{ji} - 2N(A_\rho)_{ji}h_{ij} + \cdots \\
&= 2Nc_{ij}\left|(A_\rho)_{ij} - \frac{h_{ij}}{c_{ij}}\right|^2 - 2N\frac{|h_{ij}|^2}{c_{ij}} + \cdots, \text{ where} && \text{(C.46)} \\
c_{ij} &= (S_\rho)_{ii} + (S_\rho)_{jj}, \quad h_{ij} = \frac{1}{2}(T_\rho)_{ij} - \left(\sum_{k\neq i}(S_\rho)_{ik}(A_\rho)_{kj} + \sum_{k\neq j}(S_\rho)_{kj}(A_\rho)_{ik}\right).
\end{aligned}$$

Therefore, the nondiagonal part $(A_\rho)_{ij}$ is updated as

$$(A_\rho)_{ij} = \frac{X_{ij} + iY_{ij}}{\sqrt{4Nc_{ij}}} + \frac{h_{ij}}{c_{ij}}, \tag{C.47}$$

This completes the algorithm for the heat bath algorithm for the bosonic IIB matrix model with the Chern-Simons term. In order to corroborate the legitimacy of the code, it is useful to exploit the following relation coming from the Schwinger-Dyson equation

$$0 = \int d^d A \sum_{a=1}^{N^2-1} \sum_{\mu=1}^{d} \frac{\partial}{\partial A_\mu^a} [Tr(t^a A_\mu) e^{-S}]. \tag{C.48}$$

Here, t^a is the basis of the $SU(N)$ Lie algebra and satisfies the following identities:

$$Tr(t^a t^b) = \delta^{ab}, \quad \sum_{a=1}^{N^2-1} (t^a)_{ij} (t^a)_{kl} = \delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}. \tag{C.49}$$

The matrices A_μ are expanded in terms of t^a as $A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a t^a$. This leads to the following identities:

$$\begin{aligned}\sum_{a=1}^{N^2-1} Tr(t^a A) Tr(t^a B) &= \sum_{a=1}^{N^2-1} A_{ji} B_{lk} (t^a)_{ij} (t^a)_{kl} = A_{ji} B_{lk} (\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}) \\ &= Tr(AB) - \frac{1}{N} TrATrB = TrAB. \end{aligned} \tag{C.50}$$

At the last equality, we utilize the tracelessness of the $SU(N)$ Lie algebra (namely, $TrA = TrB = 0$).

Recalling the fact that

$$\frac{\partial S}{\partial A_\mu^a} = -NTr(t^a [A_\nu, [A_\mu, A_\nu]]) - \lambda d N \epsilon_{\mu\nu_1\cdots\nu_{d-1}} Tr(t^a A_{\nu_1} A_{\nu_2} \cdots A_{\nu_{d-1}}),$$

we rewrite the Schwinger-Dyson equation (C.48) as

$$\begin{aligned} 0 &= \int d^d A \sum_{a=1}^{N^2-1} [Tr(t^a t^a) d e^{-S} + NTr(t^a A_\mu) Tr(t^a [A_\nu, [A_\mu, A_\nu]]) e^{-S} + \\ &\qquad + \lambda d N \epsilon_{\mu\nu_1\cdots\nu_{d-1}} Tr(t^a A_\mu) Tr(t^a A_{\nu_1} A_{\nu_2} \cdots A_{\nu_{d-1}}) e^{-S}] \\ &= \int d^d A \left[d(N^2-1) + NTr\left([A_\mu, A_\nu]^2 + \lambda \epsilon_{\mu_1\cdots\mu_d} A_{\mu_1} A_{\mu_2} \cdots A_{\mu_d} \right) \right] e^{-S} \\ &= \left(\int d^d A e^{-S} \right) \times \left(d(N^2-1) + \frac{N \int d^d A Tr\left([A_\mu, A_\nu]^2 + \lambda \epsilon_{\mu_1\cdots\mu_d} A_{\mu_1} A_{\mu_2} \cdots A_{\mu_d} \right) e^{-S}}{\int d^d A e^{-S}} \right). \end{aligned} \tag{C.51}$$

This implies that the vacuum expectation value of the following quantity is analytically computed as

$$-\frac{1}{N}\langle Tr[A_\mu, A_\nu]^2 \rangle - \frac{\lambda}{N} \langle \epsilon_{\mu_1\cdots\mu_d} Tr A_{\mu_1} A_{\mu_2} \cdots A_{\mu_d} \rangle = d(1 - \frac{1}{N^2}). \tag{C.52}$$

C.5 Jackknife (binning) method

Finally, we introduce the jackknife method to estimate the errorbar of the quantities obtained by the numerical simulation. Now, we have a sample of independent measurements of a primary quantity A. The measured samples are

$$A_1, A_2, \cdots, A_N. \tag{C.53}$$

The sample average is simply defined as

$$\bar{A} = \frac{1}{N} \sum_{s=1}^{N} A_s. \tag{C.54}$$

Next, we define the *jackknife average*. We exclude *one* of the sets $\{A_n\}$ in taking the average:

$$A_{(J)s} = \frac{1}{N-1} \sum_{r \neq s} A_r. \tag{C.55}$$

The variance of the jackknife estimators can be obtained as

$$\sigma^2_{(J)\bar{A}} = \frac{N-1}{N}\sum_{s=1}^{N}(A_{(J)s} - \bar{A})^2. \tag{C.56}$$

Note that the average $\bar{A}_{(J)s}$ is trivially equivalent to the average $\bar{A}_s$. We can verify $\sigma_{(J)\bar{A}} = \sigma_{\bar{A}}$ in the following way.

$$\begin{aligned}
\sigma^2_{(J)\bar{A}} &= \frac{N-1}{N}\sum_{s=1}^{N}\left(\bar{A}^2 - \frac{2}{(N-1)}\bar{A}(N\bar{A} - A_s) + \frac{1}{(N-1)^2}(N\bar{A} - A_s)^2\right). \\
&= \frac{N-1}{N}\left[N\bar{A}^2 - \frac{2N}{(N-1)}(N\bar{A}^2 - \bar{A}^2) + \frac{1}{(N-1)^2}\left((N^3\bar{A}^2 - 2N^2\bar{A}^2) + \sum_{s=1}^{N}(A_s^2)\right)\right] \\
&= \bar{A}^2\left(-(N-1) + \frac{N(N-2)}{(N-1)}\right) + \frac{1}{(N-1)N}\sum_{s=1}^{N}(A_s^2) \\
&= \frac{1}{(N-1)}\left(\frac{1}{N}\sum_{s=1}^{N}(A_s^2) - \bar{A}^2\right) = \sigma^2_{\bar{A}}.
\end{aligned} \tag{C.57}$$

Now, $\sigma_{\bar{A}}$ is an *unbiased estimator* of the *variance*:

$$E\left(\frac{1}{N-1}\sum_{s=1}^{N}(A_i - \bar{A})^2\right) = \frac{1}{N-1}\sum_{s=1}^{N}\frac{N-1}{N}\sigma^2 = \sigma^2. \tag{C.58}$$

Next, we explain the jackknife binning. This method is utilized when A_s's have correlations. Especially, the configurations generated by the Markov process are not statistically independent. This poses the problem of the "autocorrelation". In coping with this problem, we use the jackknife binning. We consider the following bins:

$$\underbrace{\underbrace{\spadesuit\spadesuit\spadesuit\spadesuit}_{\text{one bin with } n = \frac{N}{M} \text{ elements}}, \underbrace{\spadesuit\spadesuit\spadesuit\spadesuit}_{\text{one bin}}, \cdots, \underbrace{\spadesuit\spadesuit\spadesuit\spadesuit}_{\text{one bin}}}_{M \text{ bins}}. \tag{C.59}$$

Suppose one bin contains n elements, and there are M bins. Then, we take the average for *each bin* as follows:

$$A_{(J)b} = \frac{1}{N-n}\sum_{s\notin(\text{Bin})_b} A_s, \tag{C.60}$$

where b runs over $1, 2, \cdots, M$. We take the variance for these $A_{(J)b}$:

$$\sigma^2_{(Bin)}(n) = \frac{M-1}{M}\sum_{b=1}^{M}(A_{(J)b} - \bar{A}_{(J)b})^2. \tag{C.61}$$

Note that $\bar{A}_{(J)b} = \bar{A}$. This variance depends on the number of the elements in one bin n, and this variance increases with n and we finally see the plateau.

To illustrate this process, let us take one simple example. We have the data

$$A_1 = 1, \quad A_2 = 2, \quad \cdots, \quad A_{20} = 20, \tag{C.62}$$

and we construct the bin as

$$\{A_1, \cdots, A_4\}, \{A_5, \cdots, A_8\}, \cdots, \{A_{17}, \cdots, A_{20}\}. \tag{C.63}$$

In this case, $N=20$, $n=4$ and $M=5$. Namely, we define the average for each bin as

$$
\begin{aligned}
A_{(J)b=1} &= \frac{(5+6+7+8)+(9+10+11+12)+(13+14+15+16)+(17+18+19+20)}{20-4} = 12.5, \\
A_{(J)b=2} &= \frac{(1+2+3+4)+(9+10+11+12)+(13+14+15+16)+(17+18+19+20)}{20-4} = 11.5, \\
A_{(J)b=3} &= \frac{(1+2+3+4)+(5+6+7+8)+(13+14+15+16)+(17+18+19+20)}{20-4} = 10.5, \\
A_{(J)b=4} &= \frac{(1+2+3+4)+(5+6+7+8)+(9+10+11+12)+(17+18+19+20)}{20-4} = 9.5, \\
A_{(J)b=5} &= \frac{(1+2+3+4)+(5+6+7+8)+(9+10+11+12)+(13+14+15+16)}{20-4} = 8.5.
\end{aligned}
$$

The average is of course $\bar{A}_{(J)b} = 10.5$. The variance is computed as

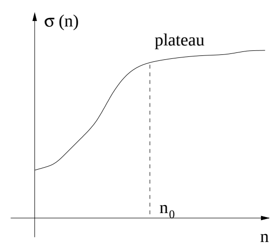

Figure 24: The plateau emerges as we increase the bin size.

$$
\begin{aligned}
\sigma^2_{(Bin)} &= \frac{5-1}{5}\left[(12.5-10.5)^2+(11.5-10.5)^2+(10.5-10.5)^2+(9.5-10.5)^2+(8.5-10.5)^2\right] \\
&= \frac{4}{5}\times 10 = 8. \qquad \text{(C.64)}
\end{aligned}
$$

The process in taking the plateau is too delicate to delegate to a computer, and we usually judge the plateau by hand.

References

[1] P. Ginsparg and G. W. Moore, “Lectures On 2-D Gravity And 2-D String Theory,” arXiv:hep-th/9304011.

[2] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, “2-D Gravity and random matrices,” Phys. Rept. **254**, 1 (1995) [arXiv:hep-th/9306153].

[3] A. H. Chamseddine and A. Connes, “The spectral action principle,” Commun. Math. Phys. **186**, 731 (1997) [arXiv:hep-th/9606001].

[4] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, “M theory as a matrix model: A conjecture,” Phys. Rev. D **55**, 5112 (1997) [arXiv:hep-th/9610043].

[5] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, “A large-N reduced model as superstring,” Nucl. Phys. B **498**, 467 (1997) [arXiv:hep-th/9612115].

[6] R. Dijkgraaf, E. Verlinde and H. Verlinde, “Matrix string theory,” Nucl. Phys. B **500**, 43 (1997) [arXiv:hep-th/9703030].

[7] M. Fukuma, H. Kawai, Y. Kitazawa and A. Tsuchiya, “String field theory from IIB matrix model,” Nucl. Phys. B **510**, 158 (1998) [arXiv:hep-th/9705128].

[8] S. Hirano and M. Kato, “Topological matrix model,” Prog. Theor. Phys. **98**, 1371 (1997) [arXiv:hep-th/9708039].

[9] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[10] J. Castelino, S. M. Lee and W. I. Taylor, “Longitudinal 5-branes as 4-spheres in matrix theory,” Nucl. Phys. B **526**, 334 (1998) [arXiv:hep-th/9712105].

[11] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, “Space-time structures from IIB matrix model,” Prog. Theor. Phys. **99**, 713 (1998) [arXiv:hep-th/9802085].

[12] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[13] T. Hotta, J. Nishimura and A. Tsuchiya, “Dynamical aspects of large N reduced models,” Nucl. Phys. B **545**, 543 (1999) [arXiv:hep-th/9811220].

[14] S. Iso and H. Kawai, “Space-time and matter in IIB matrix model: Gauge symmetry and diffeomorphism,” Int. J. Mod. Phys. A **15** (2000) 651 [arXiv:hep-th/9903217].

[15] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[16] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, A. Tsuchiya and T. Tada, “IIB matrix model,” Prog. Theor. Phys. Suppl. **134**, 47 (1999) [arXiv:hep-th/9908038].

[17] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, “Noncommutative Yang-Mills in IIB matrix model,” Nucl. Phys. B **565**, 176 (2000) [arXiv:hep-th/9908141].

[18] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP **9909**, 032 (1999) [arXiv:hep-th/9908142].

[19] L. Smolin, “M theory as a matrix extension of Chern-Simons theory,” Nucl. Phys. B **591**, 227 (2000) [arXiv:hep-th/0002009].

[20] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, “Large N dynamics of dimensionally reduced 4D SU(N) super Yang-Mills theory,” JHEP **0007**, 013 (2000) [arXiv:hep-th/0003208].

[21] E. Bergshoeff and A. Van Proeyen, “The many faces of OSp(1|32),” Class. Quant. Grav. **17**, 3277 (2000) [arXiv:hep-th/0003261].

[22] J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, “Monte Carlo studies of the IIB matrix model at large N,” JHEP **0007**, 011 (2000) [arXiv:hep-th/0005147].

[23] L. Smolin, “The cubic matrix model and a duality between strings and loops,” arXiv:hep-th/0006137.

[24] P. Austing and J. F. Wheater, “The convergence of Yang-Mills integrals,” JHEP **0102**, 028 (2001) [arXiv:hep-th/0101071].

[25] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, “Noncommutative gauge theory on fuzzy sphere from matrix model,” Nucl. Phys. B **604**, 121 (2001) [arXiv:hep-th/0101102].

[26] T. Azuma, S. Iso, H. Kawai and Y. Ohwashi, “Supermatrix models,” Nucl. Phys. B **610**, 251 (2001) [arXiv:hep-th/0102168].

[27] T. Azuma, “Investigation of matrix theory via super Lie algebra,” arXiv:hep-th/0103003.

[28] P. Austing and J. F. Wheater, “Convergent Yang-Mills matrix theories,” JHEP **0104**, 019 (2001) [arXiv:hep-th/0103159].

[29] Y. Kimura, “Noncommutative gauge theories on fuzzy sphere and fuzzy torus from matrix model,” Prog. Theor. Phys. **106**, 445 (2001) [arXiv:hep-th/0103192].

[30] L. Smolin, “The exceptional Jordan algebra and the matrix string,” arXiv:hep-th/0104050.

[31] S. Ramgoolam, "On spherical harmonics for fuzzy spheres in diverse dimensions," Nucl. Phys. B **610**, 461 (2001) [arXiv:hep-th/0105006].

[32] T. Azuma and H. Kawai, "OPE between the energy momentum tensor and the Wilson loop in N = 4 Super-Yang-Mills theory," Prog. Theor. Phys. **106**, 255 (2001) [arXiv:hep-th/0106063].

[33] S. Bal and H. Takata, "Interaction between two fuzzy spheres," Int. J. Mod. Phys. A **17**, 2445 (2002) [arXiv:hep-th/0108002].

[34] P. Austing, "Yang-Mills matrix theory," arXiv:hep-th/0108128.

[35] Y. Ohwashi, "E(6) matrix model," Prog. Theor. Phys. **108**, 755 (2002) [arXiv:hep-th/0110106].

[36] J. Nishimura and F. Sugino, "Dynamical generation of four-dimensional space-time in the IIB matrix model," JHEP **0205**, 001 (2002) [arXiv:hep-th/0111102].

[37] M. Bagnoud, L. Carlevaro and A. Bilal, "Supermatrix models for M-theory based on osp(1|32,R)," Nucl. Phys. B **641**, 61 (2002) [arXiv:hep-th/0201183].

[38] T. Azuma and H. Kawai, "Matrix model with manifest general coordinate invariance," Phys. Lett. B **538**, 393 (2002) [arXiv:hep-th/0204078].

[39] H. Kawai, S. Kawamoto, T. Kuroki, T. Matsuo and S. Shinohara, "Mean field approximation of IIB matrix model and emergence of four dimensional space-time," Nucl. Phys. B **647**, 153 (2002) [arXiv:hep-th/0204240].

[40] Y. Kimura, "Noncommutative gauge theory on fuzzy four-sphere and matrix model," Nucl. Phys. B **637**, 177 (2002) [arXiv:hep-th/0204256].

[41] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," Nucl. Phys. B **644**, 3 (2002) [arXiv:hep-th/0206255].

[42] R. Dijkgraaf and C. Vafa, "On geometry and matrix models," Nucl. Phys. B **644**, 21 (2002) [arXiv:hep-th/0207106].

[43] S. Ramgoolam, "Higher dimensional geometries related to fuzzy odd-dimensional spheres," JHEP **0210**, 064 (2002) [arXiv:hep-th/0207111].

[44] Y. Kitazawa, "Matrix models in homogeneous spaces," Nucl. Phys. B **642**, 210 (2002) [arXiv:hep-th/0207115].

[45] R. Dijkgraaf and C. Vafa, "A perturbative window into non-perturbative physics," arXiv:hep-th/0208048.

[46] T. Azuma and M. Bagnoud, "Curved-space classical solutions of a massive supermatrix model," Nucl. Phys. B **651**, 71 (2003) [arXiv:hep-th/0209057].

[47] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, "Chiral rings and anomalies in supersymmetric gauge theory," JHEP **0212**, 071 (2002) [arXiv:hep-th/0211170].

[48] H. Kawai, S. Kawamoto, T. Kuroki and S. Shinohara, "Improved perturbation theory and four-dimensional space-time in IIB matrix model," Prog. Theor. Phys. **109**, 115 (2003) [arXiv:hep-th/0211272].

[49] F. Cachazo, N. Seiberg and E. Witten, "Phases of N = 1 supersymmetric gauge theories and matrices," JHEP **0302**, 042 (2003) [arXiv:hep-th/0301006].

[50] Y. Kimura, "On higher dimensional fuzzy spherical branes," Nucl. Phys. B **664**, 512 (2003) [arXiv:hep-th/0301055].

[51] T. A. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, "Quantum corrections on fuzzy sphere," Nucl. Phys. B **665**, 520 (2003) [arXiv:hep-th/0303120].

[52] Y. J. Zhong and W. H. Huang, "Quantum energy of fuzzy sphere: Gaussian variational method," arXiv:hep-th/0303196.

[53] F. Cachazo, N. Seiberg and E. Witten, "Chiral Rings and Phases of Supersymmetric Gauge Theories," JHEP **0304**, 018 (2003) [arXiv:hep-th/0303207].

[54] H. Kawai, T. Kuroki and T. Morita, "Dijkgraaf-Vafa theory as large-N reduction," Nucl. Phys. B **664**, 185 (2003) [arXiv:hep-th/0303210].

[55] T. Imai, Y. Kitazawa, Y. Takayama and D. Tomino, "Effective actions of matrix models on homogeneous spaces," Nucl. Phys. B **679**, 143 (2004) [arXiv:hep-th/0307007].

[56] D. Tomino, "N = 2 3d-matrix integral with Myers term," arXiv:hep-th/0309264.

[57] P. Austing and J. F. Wheater, "Adding a Myers term to the IIB matrix model," JHEP **0311**, 009 (2003) [arXiv:hep-th/0310170].

[58] H. Kawai, T. Kuroki and T. Morita, "Supersymmetric large-N reduced model with multiple matter," arXiv:hep-th/0312026.

[59] T. Imai and Y. Takayama, "Stability of fuzzy $S^2 \times S^2$ geometry in IIB matrix model," arXiv:hep-th/0312241.

[60] T. Azuma, S. Bal, K. Nagao and J. Nishimura, "Nonperturbative studies of fuzzy spheres in a matrix model with the Chern-Simons term," arXiv:hep-th/0401038.

[61] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, "Planar Diagrams," Commun. Math. Phys. **59**, 35 (1978).

[62] D. Friedan, "Introduction To Polyakov's String Theory," EFI-82-50-CHICAGO *To appear in Proc. of Summer School of Theoretical Physics: Recent Advances in Field Theory and Statistical Mechanics, Les Houches, France, Aug 2-Sep 10, 1982.*

[63] T. Eguchi and H. Kawai, "Reduction Of Dynamical Degrees Of Freedom In The Large N Gauge Theory," Phys. Rev. Lett. **48**, 1063 (1982).

[64] H. Kawai and Y. Okamoto, "Entropy Of Planar Random Surfaces On The Lattice," Phys. Lett. B **130**, 415 (1983).

[65] A. Gonzalez-Arroyo and M. Okawa, "The Twisted Eguchi-Kawai Model: A Reduced Model For Large N Lattice Gauge Theory," Phys. Rev. D **27**, 2397 (1983).

[66] A. Gonzalez-Arroyo and C. P. Korthals Altes, Phys. Lett. B **131**, 396 (1983).

[67] K. Konishi, "Anomalous Supersymmetry Transformation Of Some Composite Operators In Sqcd," Phys. Lett. B **135**, 439 (1984).

[68] K. i. Konishi and K. i. Shizuya, "Functional Integral Approach To Chiral Anomalies In Supersymmetric Gauge Theories," Nuovo Cim. A **90**, 111 (1985).

[69] S. R. Das, "Some Aspects Of Large N Theories," Rev. Mod. Phys. **59**, 235 (1987).

[70] F. David, "Planar Diagrams, Two-Dimensional Lattice Gravity And Surface Models," Nucl. Phys. B **257**, 45 (1985).

[71] T. W. Körner, "Fourier Analysis" Cambridge University Press (1988) Chap. 6

[72] J. Distler and H. Kawai, "Conformal Field Theory And 2-D Quantum Gravity Or Who's Afraid Of Joseph Liouville?," Nucl. Phys. B **321**, 509 (1989).

[73] E. Witten, "Quantum Field Theory And The Jones Polynomial," Commun. Math. Phys. **121**, 351 (1989).

[74] E. Brezin and V. A. Kazakov, "Exactly Solvable Field Theories Of Closed Strings," Phys. Lett. B **236**, 144 (1990).

[75] S. Shinohara, *Soryushiron Kenkyu* **101** (2000) 111
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